INTERFERENCE MANAGEMENT IN WIRELESS NETWORKS

BY

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DISSERTATION

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ABSTRACT

Interference management in wireless networks has emerged as an important task in order to meet the increased demand for data. In this thesis, interference management through cooperative transmission in the downlink is studied for a cellular network. Degrees of freedom (DoF) gains are first studied in a hexagonal sectored cellular network with cooperative transmission under a *backhaul load* constraint that limits the average number of messages that can be delivered from a centralized controller to basestation transmitters. The backhaul load is defined as the sum of all the messages available at all the transmitters per channel use, normalized by the number of users. Using insights from the analysis of Wyner's linear interference network, the results are extended to the more practical hexagonal sectored cellular network, and coding schemes based on cooperative zero-forcing are shown to deliver significant DoF gains. It is established that by allowing for cooperative transmission and a flexible message assignment that is constrained only by an average backhaul load, one can deliver the rate gains promised by *information-theoretic upper bounds* with practical one-shot schemes that incur little or no additional load on the backhaul. Finally, useful upper bounds on the per user DoF for schemes based on cooperative zero-forcing are presented for the average backhaul load constraint, and an optimization framework is formulated for the general converse problem.

Degrees of freedom (DoF) gains through cooperative transmission are then studied in the downlink of a two-layered heterogeneous network with macro basestations (MBs), small-cell basestations (SBs) that act as *half-duplex* analog relays, and mobile terminals (MTs). The first layer is a wireless *backhaul* layer between MBs and SBs, and the second layer is the *transmission* layer between SBs and MTs. The two layers use the same time/frequency resources for communication, limiting the maximum per user degrees of freedom (puDoF) to half, due to the half-duplex nature of the SBs. A linear

network is first considered, and it is established that the optimal puDoF can be achieved by cooperation with sufficient antennas. The proposed schemes are simple zero-forcing schemes that achieve cooperation without overloading the backhaul. Cooperation is implemented by sending an appropriate linear combination of users' messages from the MBs to the SBs that zero-force interference at the MTs. The achievable schemes exploit the half-duplexity of the SBs and schedule the SBs and MTs to be active in different time-slots in a smart manner to reduce interference. These results are then extended to a more realistic hexagonal network, and it is shown that the optimal puDoF of half can be approached using only zero-forcing schemes, without using interference alignment.

Interference management is then considered through the design of an efficient algorithm in a decentralized uncoordinated spectrum sharing system. A multi-user multi-armed bandit (MAB) framework is used to develop algorithms for uncoordinated spectrum access. The number of users is assumed to be unknown to each user. A stochastic setting is first considered, where the rewards on a channel are the same for each user. In contrast to prior work, it is assumed that the number of users can possibly exceed the number of channels, and that rewards can be non-zero even under collisions. The proposed algorithm consists of an estimation phase and an allocation phase. It is shown that if every user adopts the algorithm, the systemwide regret is sub-linear over a horizon of time T. The regret guarantees hold for any number of users and channels; in particular, they hold even when the number of users is less than the number of channels. Next, an adversarial multi-user MAB framework is considered, where the rewards on the channels are user-dependent. It is assumed that the number of users is less than the number of channels, and that the users receive zero reward on collision. The proposed algorithm combines the Exp3.P algorithm developed in prior work for single-user adversarial bandits with a collision resolution mechanism to achieve sub-linear regret. It is shown that if every user employs the proposed algorithm, the systemwide regret is $O(T^{\frac{3}{4}})$ over a horizon of time T. The algorithms in both stochastic and adversarial scenarios are extended to the dynamic case where the number of users in the system evolves over time and are shown to lead to sub-linear regret.

To my parents, for their love and support

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CHAPTER 1

INTRODUCTION

There has been a rapid growth in the usage of wireless devices in the past few years, and it has been predicted that global mobile data traffic will increase sevenfold between 2016 and 2021 [1]. Thus, there is an increased demand for data in wireless networks.

In a cellular network, interference due to neighboring cells is a major factor that limits the rates of users. Thus, interference management in cellular networks has emerged as an important task in order to meet the increased demand for data. Cooperation among basestations or mobile users in the cellular network has emerged as one of the important technologies for managing interference [2].

To meet the increasing demand for mobile traffic, heterogeneous networks are envisioned to be a key component of future cellular networks [3]. Heterogeneous networks enable flexible and low-cost deployments and provide a uniform broadband experience to users anywhere in the network [4]. Heterogeneous networks are designed with the addition of basestations in the networks in a hierarchical manner, going from macro to micro to pico to femto basestations, making the interference management problem more difficult [5]. Managing interference in such heterogeneous networks is crucial in order to achieve higher data rates for the users.

Another way to meet the demand for increased data is by treating frequency spectrum as a shared resource. Dynamic spectrum access has emerged to avoid spectrum under-utilization. An important aspect in such systems is the design of a dynamic access scheme that avoids or minimizes interference among the users in the system.

In this work, we focus on interference management through cooperative transmission in the downlink of a hexagonal sectored cellular network first and then for a heterogeneous network. We also study a decentralized spectrum sharing system without any hierarchy among the users and develop schemes to manage interference among the users to ensure efficient spectrum access without coordination.

1.1 Cooperative Transmission in Downlink

We wish to understand the fundamental limits of interference management using cooperative transmission in the cellular downlink and propose practical schemes that can approach these limits. We use the degrees of freedom metric to analyze the limits of the cellular network.

Degrees of freedom is a high SNR approximation of the capacity of the network that captures the number of interference-free sessions in the network at high SNR. Our focus on the DoF criterion is justified by the fact that it is useful to capture roughly the available capacity as a fraction of the capacity of an interference-free network consisting of point-to-point links. Two major advantages of the DoF criterion are as follows: (i) it is easy to analyze, and in many cases, the problem of finding an information theoretic upper bound or converse reduces to a straightforward combinatorial problem; and (ii) it captures the effect of interference, while circumventing the difficulties in analysis introduced by the additive Gaussian noise at the receivers. The DoF of a point-to-point link with white Gaussian noise is unity, and this is the reference benchmark for any given user's rate in an interference network, i.e., the per user DoF is at most one. A comprehensive overview of DoF analysis for interference management can be found in [5].

A major advance in the theoretical analysis of interference management in large wireless networks took place with the introduction of asymptotic interference alignment (IA) in [6]. IA relies on signaling over a number of time slots (*symbol extension*) that goes to infinity in order to enable the achievability of a per user DoF of $\frac{1}{2}$ in a fully connected interference network. However, the gains offered by IA are considered to be infeasible in practice, and a major reason for the infeasibility is the excessive requirement on the length of the symbol extension, which would lead to impractical delays.

We focus on more practical models than the fully connected model and show that the promised gains of interference alignment can be achieved with one-shot coding schemes that do not require symbol extension, using cooperative transmission. We show this in the downlink of a sectored hexagonal cellular network, first in a single-layer locally connected interference network in Chapter 2, and then in a heterogeneous network modeled as a two-layered locally connected interference network in Chapter 3.

1.2 Dynamic Spectrum Access

The existing spectrum management paradigm treats frequency spectrum as a fixed commodity and leads to spectrum under-utilization. Dynamic spectrum access has emerged as a useful strategy to increase spectrum utilization. In existing literature, dynamic spectrum access is largely focused on the primary/secondary user paradigm where secondary users need to detect vacant spectrum when available and vacate occupied spectrum when a primary user wants to transmit. Coordination among the users is assumed, and the distinction between primary and secondary users itself provides critical structural knowledge upon which coordination can be established.

We focus on a different type of spectrum sharing system in which multiple users attempt to access a wideband spectrum. There is no distinction between users, and the users do not coordinate with each other. The collective performance across all users is more important than that of individual users. Thus the burden of ensuring fair and efficient spectrum sharing is on all users. This is in contrast to the typical primary/secondary user paradigm in which secondary users bear the responsibility for ensuring priority-based spectrum sharing.

In Chapter 4, we model the system as a multi-user multi-armed bandit (MAB) problem with the channels corresponding to the arms. The interference in the system is captured through the reward observed by each user, and hence the systemwide regret. We consider a stochastic model first and then an adversarial model for the rewards. In the stochastic setting, we assume that the number of users can be greater than the number of channels. We further assume that when multiple users choose the same channel, the reward observed is inversely proportional to the number of users on the channel. In the adversarial setting, we assume that the number of users is less than the number of channels, and that the reward is zero when there is interference, i.e., when more than one user chooses the same channel. Our goal is to have efficient channel access by managing interference in the system by means of a decentralized policy at each user that achieves sub-linear regret with time.

CHAPTER 2

HEXAGONAL CELLULAR NETWORK

In this chapter, we explore the potential degrees of freedom gains of cooperative transmission in the hexagonal sectored cellular network with no intra-cell interference, under an average backhaul load constraint. In particular, we show that cooperative transmission can be used to achieve significant DoF gains without requiring extra backhaul capacity.

The DoF gain offered by cooperative transmission[1] in Wyner's linear interference networks was studied in [7], for the special case where each message is available at the transmitter with the same index as well as $M-1$ succeeding transmitters. The asymptotic limit of the per user DoF as the number of users goes to infinity was shown to be $\frac{M}{M+1}$. An asymptotic per user DoF of $\frac{2M-1}{2M}$ was achieved using a smarter message assignment in [8]. In the proposed scheme of [8], each message is assigned to the transmitter with the same index as well as $M-1$ other transmitters. However, unlike the assignment of [7], in [8] the choice of the $M-1$ other transmitters is not simply the succeeding $M-1$ transmitters. In [9], it is shown that under a cooperation order constraint that limits the number of transmitters at which each message can be available by M, the asymptotic per user DoF is $\frac{2M}{2M+1}$ and is achieved by a *flexible* assignment of messages to transmitters where it is not necessary to assign each message to the transmitter with the same index. The DoF gains discussed in [7], [8] and [9] are achieved by a simple signaling scheme that relies only on zero-forcing transmit beamforming.

The maximum transmit set size constraint of M is not met tightly for all messages in the optimal message assignment scheme presented in [9]. In [10], the *backhaul load* constraint B is considered that is more general and relevant to many scenarios of practical significance. The backhaul load constraint B is defined as the ratio between the sum of the transmit set sizes for all the messages and the number of users. In other words, the transmit set size varies

[1]Also called Coordinated Multi-Point (CoMP) transmission [2].

across the messages, while maintaining a constraint on the average transmit set size of B. The asymptotic per user DoF for the Wyner model is shown to be $\frac{4B-1}{4B}$ in [10], which is larger than the per user DoF of $\frac{2M}{2M+1}$ obtained with the more stringent per message transmit set size constraint of $M = B$. The identified optimal scheme relies only on zero-forcing beamforming at the transmitters, and an asymmetric or unbalanced assignment of messages, with some messages being assigned to more than B transmitters and others being assigned to fewer than B transmitters.

We apply these insights to the more practical hexagonal sectored cellular model. We show that with cooperative transmission based on zero-forcing beamforming with asymmetric assignment of messages under an integer backhaul load constraint of B, it is possible to achieve a per user DoF of $\frac{2B}{3B+1}$. We also show that under restriction to zero-forcing schemes, the asymptotic per user DoF is upper bounded by $\frac{5+B}{10}$ for $B < 5$ (Theorem 6), and formulate the general problem of finding the maximum per user DoF under restriction to zero-forcing schemes as an optimization problem. We emphasize that a per user DoF of $\frac{1}{2}$ is achievable with simple zero-forcing schemes and $B = 1$, i.e., with no additional backhaul load. On the other hand, we show that if cooperative transmission is not allowed ($M = 1$), then a per user DoF of $\frac{1}{2}$ is the optimal value, and cannot be obtained by simple interference avoidance schemes. This shows that simple one-shot zero-forcing beamforming combined with non-uniform message assignments can be used to achieve significant gains in the per user DoF, while maintaining a low average backhaul load. Degrees of freedom gains in the hexagonal cellular downlink using CoMP transmission was also considered in [11], where the transmitting basestations cooperate by exchanging quantized dirty paper coded signals. However, such a scheme can suffer from propagation delays due to successive encoding.

2.1 System Model and Notation

We use the standard model for the K-user interference channel with single-antenna transmitters and receivers,

$$Y_i = \sum_{j \in \mathcal{N}_i} H_{i,j} X_j + Z_i, \tag{2.1}$$

where X_j denotes the signal transmitted by transmitter j under an average transmit power constraint, Z_i denotes the additive white Gaussian noise at receiver i, $H_{i,j}$ denotes the channel gain coefficient from transmitter j to receiver i, and $\mathcal{N}_i$ denotes the set of transmitters that can be heard at receiver i (neighbors in the connectivity graph including itself). All channel coefficients that are not identically zero are assumed to be drawn from a continuous joint distribution. Finally, it is assumed that global channel state information is available at all transmitters and receivers.

2.1.1 Hexagonal Cellular Network

This is a sectored K user cellular network with three sectors per cell as shown in Figure 2.1(a). We assume a local interference model, where the interference at each receiver is only due to the basestations in the neighboring sectors in adjacent cells. It is assumed that the sectors belonging to the same cell do not interfere with each other, the justification being that the interference power due to sectors in the same cell is usually far lower than the interference from out-of-cell users located in the sector's line of sight.

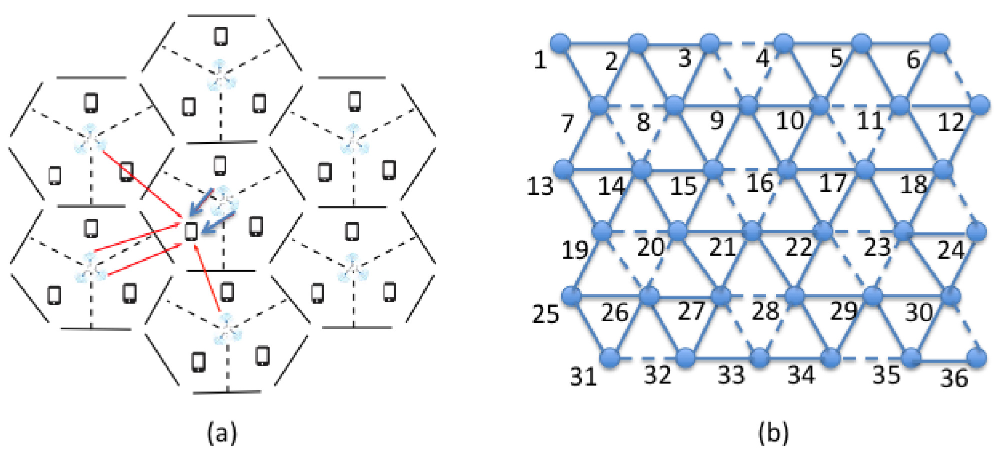


Figure 2.1: (a) Cellular network and (b) connectivity graph. The dotted lines in (b) represent the interference between sectors belonging to the same cell.

Connectivity graph

The cellular model is represented by an undirected connectivity graph $G(V, E)$ shown in Figure 2.1(b) where each vertex $u \in V$ corresponds to a transmitter-receiver pair. For any node a, the transmitter, receiver and intended message (word) corresponding to the node are denoted by T_a, R_a and W_a, respectively. An edge $e \in E$ between two vertices $u, v \in V$ corresponds to a channel existing between the transmitter at u and the receiver at v, and vice-versa. The dotted lines denote interference between sectors that belong to the same cell, and is ignored in our model. For any node a, $\mathcal{N}_a$ denotes the set of nodes adjacent to node a and that includes node a. To simplify the presentation, without much loss of generality, we consider only $K-$user networks where $\sqrt{K}$ is an integer, and nodes are numbered as in Figure 2.1(b). (In the figure, $\sqrt{K} = 6$.) Since we are studying the performance in the asymptotic limit of the number of users, the assumption is not restrictive.

We formally define the connectivity graph $G(V, E)$ using Eisenstein integers similar to [11].

Definition 1. *(Eisenstein integers) : Eisenstein integers $\mathbb{Z}[\omega]$ are numbers of the form $a + b\omega$ where $\omega = \frac{1}{2}(-1 + \imath\sqrt{3})$ and $a, b \in \mathbb{Z}$, where $\imath = \sqrt{-1}$.*

Let $\mathbb{B}_r = \{z \in \mathbb{C} : |\mathrm{Re}(z)| \leq r, |\mathrm{Im}(z)| \leq \frac{\sqrt{3}r}{2}\}$. The set $\mathbb{B}_r$ denotes the Eisenstein integers enclosed in the rectangle centered at the origin with the real part bounded by r and the imaginary part bounded by $\frac{\sqrt{3}r}{2}$. Consider the following one-to-one mapping $g : V \to \mathbb{Z}[\omega] \cap \mathbb{B}_r$ between vertices of the graph and Eisenstein integers. For each $v \in V$, $g(v)$ denotes the corresponding vertex in the Eisenstein graph. Note that

$$V = \{g^{-1}(z) : z \in \mathbb{Z}[\omega] \cap \mathbb{B}_r\}. \tag{2.2}$$

Consider the function $f(a + b\omega) = (a + b) \bmod 3$. This partitions the space of Eisenstein integers into three cosets represented by $\Omega_{sq}, \Omega_{cir}, \Omega_{dia}$ corresponding to $f(z) = 0$, $f(z) = 1$ and $f(z) = 2$ for all $z \in \mathbb{Z}[\omega]$. The subscripts of $\Omega_{sq}, \Omega_{cir}, \Omega_{dia}$ correspond to the squares, circles and diamonds which are used to represent the respective cosets in Figure 2.2.

For any $z \in \mathbb{Z}[\omega] \cap \mathbb{B}_r$, we define the following triangle $\Delta(z)$:

$$\Delta(z) = \{z, z + \omega, z + \omega + 1\},$$

and the edges incident to the vertices of $\Delta(z)$ are denoted by $\mathcal{E}(\Delta(z))$ as follows:

$$\mathcal{E}(\Delta(z)) = \{(z, z+\omega), (z, z+\omega+1), (z+\omega, z+\omega+1)\}.$$

If we consider the edges $\mathcal{E}(\Delta(z))$, where $z \in \Omega_{sq}$, each node is incident to exactly two edges, and by removing these edges we have the connectivity graph in Figure 2.2, a proper representation of the hexagonal cellular network with no intra-cell interference. More precisely, without loss of generality, let $\mathcal{E}(\Delta(z))$, where $z \in \Omega_{sq}$, correspond to the intra-cell interference. Then since we ignore intra-cell interference in our model, we define the set of interfering edges in the graph as

$$E = \{(u, v) : u, v \in V \text{and } (g(u), g(v)) \in \mathcal{E}(D)\}, \tag{2.3}$$

where

$$D = \{\Delta(z) : z \in \{\Omega_{cir} \cup \Omega_{dia}\}\}.$$

Thus the interference graph is $G(V, E)$ where V is given by (2.2) and the set of edges E is given by (2.3).

2.1.2 Message Assignment

For each $i \in [K]$, let W_i be the message intended for receiver i, and $\mathcal{T}_i \subseteq [K]$ be the transmit set of receiver i, i.e., those transmitters with the knowledge of W_i. The transmitters in $\mathcal{T}_i$ cooperatively transmit the message W_i to the receiver i. A particular message assignment is denoted by $\{\mathcal{T}_i\}_{i\in[K]}$. For a particular message assignment, M denotes the maximum transmit set size and B denotes the backhaul load or the average transmit set size,

$$M = \max_i |\mathcal{T}_i|, \tag{2.4}$$

$$B = \frac{\sum_{i=1}^{K} |\mathcal{T}_i|}{K}. \tag{2.5}$$

In this work, we allow for flexible association of messages, i.e., we only restrict the size of transmit sets, without constraints on the specific set of transmitters that each message is assigned to. The case $M = 1$ corresponds

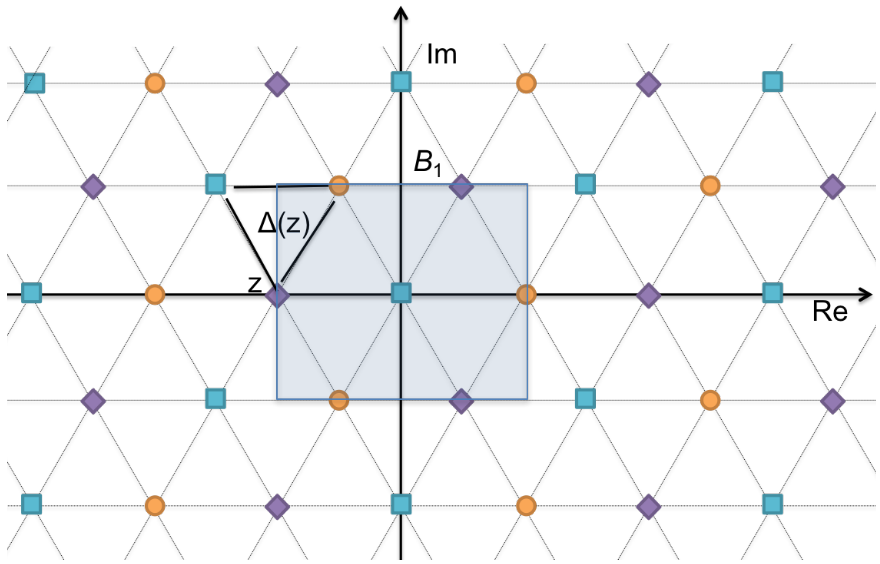


Figure 2.2: The cellular network is represented by Eisenstein integers, partitioned into three cosets $\Omega_{sq}, \Omega_{cir}, \Omega_{dia}$ represented by square, circle and diamond shaped nodes respectively. For any node z, $\Delta(z)$ represents the edges between the nodes, z, $z+\omega$ and $z+\omega+1$. $\mathbb{B}_r$ is illustrated in the figure for $r=1$. The seven nodes lying on or within the rectangle belong to the set $\mathbb{B}_1$.

to the case of no cooperation, but with possibly a flexible association of cells. The case $B=1$ corresponds to an average backhaul load of one message per transmitter, i.e., no extra backhaul load due to cooperation.

2.1.3 Zero-forcing Schemes

We consider in this work the class of *zero-forcing* schemes, where each message is either not transmitted or allocated one degree of freedom. Accordingly, every receiver is either active or inactive. An active receiver does not observe any interfering signals. For the case of no-cooperation, i.e., $M=1$, we refer to these schemes as interference avoidance schemes. The case where $M \geq 2$ corresponds to the scenario where cooperative zero-forcing can be used.

For any zero-forcing scheme, the transmit signal at the j^{th} transmitter is given by

$$X_j = \sum_{i: j \in \mathcal{T}_i} X_{j,i}, \tag{2.6}$$

where $X_{j,i}$ depends only on message W_i. Further, each message is either not transmitted or allocated one degree of freedom. More precisely, let $\tilde{Y}_j = Y_j - Z_j, \forall j \in [K]$. Then, in addition to the constraint in (2.6), it is either case that the mutual information $I(\tilde{Y}_j; W_j) = 0$ or it is the case that W_j completely determines $\tilde{Y}_j$. Note that $\tilde{Y}_j$ can be determined from W_j for the case where user j enjoys interference-free communication, and $I(W_j; \tilde{Y}_j) = 0$ for the other case where W_j is not transmitted. We say that the j^{th} receiver is *active* if and only if $I(\tilde{Y}_j; W_j) > 0$. Note that using zero-forcing transmit beamforming, if the j^{th} receiver is active, then $I(W_i; Y_j) = 0, \forall i \neq j$. Finally, we say that the j^{th} transmitter is *active* if $I\left(X_j; \{W_i : j \in \mathcal{T}_i\}\right) > 0$.

Without loss in generality, we assume that it has to be the case that if a message W_i is available at transmitter j, i.e., $j \in \mathcal{T}_i$, then the message contributes to the corresponding transmit signal, i.e., $I(W_i, X_{j,i}) > 0$. Otherwise, the message assignment could be removed without affecting the sum rate.

2.1.4 Degrees of Freedom

Let P be the average transmit power constraint at each transmitter, and let $\mathcal{W}_i$ denote the alphabet for message W_i. Then the rates $R_i(P) = \frac{\log |\mathcal{W}_i|}{n}$ are achievable if the decoding error probabilities of all messages can be simultaneously made arbitrarily small for a large enough coding block length n, and this holds for almost all channel realizations. The degrees of freedom (DoF) $d_i, i \in [K]$, is defined as $d_i = \lim_{P \to \infty} \frac{R_i(P)}{\log P}$. The DoF region $\mathcal{D}$ is the closure of the set of all achievable DoF tuples. The total DoF (η) is the maximum value of the sum of the achievable degrees of freedom, $\eta = \max_{\mathcal{D}} \sum_{i \in [K]} d_i$.

For a K-user channel, we define $\eta(K, M)$ and $\eta^{\text{avg}}(K, B)$ as the maximum achievable η over all possible message assignments satisfying the constraints (2.4) and (2.5) respectively. We define the following asymptotic quantities which capture how η scales with K.

$$\tau(M) = \lim_{K \to \infty} \frac{\eta(K, M)}{K}, \tag{2.7}$$

$$\tau^{\text{avg}}(B) = \lim_{K \to \infty} \frac{\eta^{\text{avg}}(K, B)}{K}. \tag{2.8}$$

We use the superscript zf to indicate a further restriction to zero-forcing schemes. Finally, we denote the DoF and asymptotic per user DoF for the hexagonal cellular network with subscript c.

2.2 Main Results

In this section, we investigate the per user DoF for the hexagonal sectored cellular model introduced in Section 2.1.1, using insights obtained from the analysis of linear networks. Our goal is to highlight the advantage of cooperative transmission that is based on flexible cell associations for cellular networks, by first showing that the asymptotic per user DoF is at most $\frac{1}{2}$ for the case when each message can be available at a single transmitter. Further, we show for this case that interference avoidance schemes can only be used to achieve an asymptotic per user DoF of at most $\frac{2}{5}$. On the other hand, when cooperative transmission is allowed, but the overall load on the backhaul is not increased ($B = 1$), interference avoidance schemes can be used to achieve the $\frac{1}{2}$ asymptotic per user DoF value.

We first impose the maximum transmit set size constraint of $M = 1$ in the network, i.e., a message of a cell edge mobile receiver can be assigned to any single basestation transmitter, thus leading to a flexible cell association in the cellular downlink.

Theorem 1. *For the considered hexagonal cellular network model, the following bound holds for the case of no-cooperation:*

$$\tau_c(M = 1) \leq \frac{1}{2}.$$

Proof. The proof is available in Section 2.2.2. □

This shows that using a traditional approach for interference management, the maximum asymptotic per user DoF for the considered hexagonal cellular network model is $\frac{1}{2}$. Further, the only known way this DoF value can be approached is in the limit as the length of symbol extension goes to infinity as in the asymptotic interference alignment scheme of [6].

2.2.1 Zero-forcing Schemes

In this section, we focus on cooperative zero-forcing, and interference avoidance which is a special case of zero-forcing schemes for $M = 1$. We now introduce some additional notation summarized in Table 2.1. For each node $i \in [K]$, let r_i indicate whether receiver i is active or not, i.e., $r_i = \mathbb{1}\{\text{Receiver } i \text{ is active}\}$, where $\mathbb{1}\{x\}$ is defined as

$$\mathbb{1}\{x\} = \begin{cases} 1, & \text{if } x \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, for each node $i \in [K]$, let t_i indicate whether transmitter i is active or not, i.e., $t_i = \mathbb{1}\{\text{Transmitter } i \text{ is active}\}$. We note that the sum DoF in the network is upper bounded by $\sum_{i\in[K]} r_i$. Consider the adjacency matrix A of the connectivity graph G of the network. The Edmond's matrix D is defined as

$$D_{ij} = \begin{cases} x_{ij}, & \text{if } A_{ij} = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where x_{ij} are indeterminates. We note that a bipartite graph G has a perfect

Table 2.1: Summary of notation used for zero-forcing bounds.

t_i	$\mathbb{1}\{\text{Transmitter } i \text{ is active}\}$
r_i	$\mathbb{1}\{\text{Receiver } i \text{ is active}\}$
$\mathcal{N}_i$	Set of nodes adjacent to node i including node i
$\mathcal{T}_j$	Set of transmitters containing message W_j
ρ_j	Fraction of users with messages available at exactly j transmitters
$\mathcal{V}_{\mathcal{S}}$	Set of active receivers connected to transmitters in $\mathcal{S}$
$D_{\mathcal{A},\mathcal{B}}$	Edmond's matrix for bipartite graph with $\mathcal{A}$ and $\mathcal{B}$

matching if and only if the polynomial defined by the determinant $\det(D)$ is not identically zero, i.e., D has full rank. Let $D_{\mathcal{A},\mathcal{B}}$ denote the Edmond's matrix for the bipartite graph with $\mathcal{A}$ and $\mathcal{B}$ as the two partite sets, and any pair of vertices that have the same index are connected in the bipartite graph.

For any set $\mathcal{S} \subseteq [K]$, we define $\mathcal{V}_{\mathcal{S}}$ as the set of indices of the active receivers connected to the transmitters with indices in $\mathcal{S}$. Then $\mathcal{V}_{\mathcal{S}} = \{k : k \in \mathcal{N}_i, i \in \mathcal{S} \text{ and } I(Y_k; W_k) > 0\}$.

We need the following lemma, which is an extension of a lemma from [9] for zero-forcing schemes.

Lemma 1. *Consider any zero-forcing scheme. For any message W_i, the number of active receivers connected to at least one transmitter carrying the message is no greater than the number of transmitters carrying the message,*

$$|\mathcal{V}_{\mathcal{T}_i}| \leq |\mathcal{T}_i|. \tag{2.9}$$

Furthermore, the following has to hold:

$$\text{rank}(D_{\mathcal{T}_i,\mathcal{V}_{\mathcal{T}_i}}) = |\mathcal{V}_{\mathcal{T}_i}|. \tag{2.10}$$

Proof. We note that (2.10) implies (2.9), but we include both in the theorem statement, and provide the proof of (2.9) first for clarity. The statement of (2.9) is the same as [9, Lemma 3], but we briefly explain the proof here for completeness. Since we impose the constraint $I(W_i; Y_j) = 0, \forall j \in \mathcal{V}_{\mathcal{T}_i}$, the interference seen at all receivers in $\mathcal{V}_{\mathcal{T}_i}$ has to be canceled. Also, since the probability of a zero Lebesgue measure set of channel realizations is zero, the $|\mathcal{T}_i|$ transmit signals carrying W_i cannot be designed to cancel W_i at more than $|\mathcal{T}_i| - 1$ receivers for almost all channel realizations. This implies (2.9).

Now, we note that (2.10) is equivalent to saying that there exists a matching between transmitters carrying W_i and active receivers connected to transmitters carrying W_i, and this matching *covers* all such active receivers. If this is not true while (2.9) is satisfied, then it follows from Hall's Marriage Theorem [12] that there have to be subsets $\tilde{\mathcal{T}} \subset \mathcal{T}_i$, $\tilde{\mathcal{V}} \subset \mathcal{V}_{\mathcal{T}_i}$ such that $|\tilde{\mathcal{T}}| < |\tilde{\mathcal{V}}|$ and any transmitter whose index is in $\mathcal{T}_i \backslash \tilde{\mathcal{T}}$ is not connected to any receiver in $\tilde{\mathcal{V}}$. Hence, the above argument that we used to reach (2.9) would apply if we consider the sets $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{V}}$ as the set of transmitters carrying W_i and the set of active receivers connected to them, respectively. It hence follows that (2.10) holds, and the proof is thus complete. □

We now characterize the per user DoF for any zero-forcing scheme.

Theorem 2. *For any K-user hexagonal cellular network, the maximum achievable zero-forcing DoF under an average backhaul load constraint, $\eta_c^{avg,zf}(K, B)$*

is the solution to the following optimization problem:

$$\max_{\{\mathcal{T}_j\},\{d_{ij}\}_{i,j\in[K]}} \sum_{i\in[K]}\sum_{j\in[K]} d_{ij} \tag{2.11}$$

$$s.t.\ \ d_{ij} \in \{0,1\}, \forall i,j \in [K], \tag{2.12}$$

$$d_{ij} = 0, \textit{if } i \notin \mathcal{N}_j \ \textit{ or } j \notin \mathcal{N}_i, \forall i,j \in [K], \tag{2.13}$$

$$\sum_{k\in\mathcal{N}_j} d_{kj} \leq 1, \forall j \in [K], \tag{2.14}$$

$$\sum_{k\in\mathcal{N}_j} d_{jk} \leq 1, \forall j \in [K], \tag{2.15}$$

$$d_{ij} \leq \mathbb{1}\{i \in \mathcal{T}_j\}, \forall i,j \in [K], \tag{2.16}$$

$$\frac{1}{K}\sum_{j} |\mathcal{T}_j| \leq B, \tag{2.17}$$

$$\text{rank}(D_{\mathcal{T}_j,\tilde{\mathcal{V}}_{\mathcal{T}_j}}) = |\tilde{\mathcal{V}}_{\mathcal{T}_j}|, \forall j \in [K], \tag{2.18}$$

where for any set $\mathcal{S} \subseteq [K]$, $\tilde{\mathcal{V}}_{\mathcal{S}} = \{k : k \in \mathcal{N}_i, i \in \mathcal{S} \textit{ and } \mathbb{1}\{\sum_{w\in\mathcal{N}_k} d_{wk} = 1\}\}$.

Proof. We first show that if the constraints in (2.12) - (2.18) are satisfied, then there exists a message assignment satisfying the average backhaul load constraint B, and a zero-forcing scheme based on this assignment that achieves a per user DoF of $\sum_{i\in[K]}\sum_{j\in[K]} d_{ij}$. It would follow then that $\eta_c^{\text{avg,zf}}(K,B) \geq \sum_{i\in[K]}\sum_{j\in[K]} d_{ij}$, and hence the direct part of the theorem would be proved. It follows from (2.17) that the sets $\{\mathcal{T}_j\}_{j\in[K]}$ are transmit sets satisfying the average backhaul load constraint. We now construct the zero-forcing scheme. If $d_{ij} = 1$, then we know from (2.16) that $i \in \mathcal{T}_j$ and we also know from (2.13) that transmitter j is connected to receiver i. We hence construct the transmit signal $X_{j,i}$ according to an optimal point-to-point code over an AWGN channel (see e.g., [13]) to deliver W_i to its destination. We know from (2.14) that X_j would not be used to deliver any other message than W_i. Hence, we only need to show that interference caused by any such message W_i at any active receiver can be canceled. From (2.18), we know that there is a matching between transmitters with indices in $\mathcal{T}_i$ and receivers with indices in $\mathcal{V}_{\mathcal{T}_i}$ that covers all such receivers. We hence assign a unique transmitter with an index $t \in \mathcal{T}_i \backslash \{j\}$ to each receiver with an index $r \in \mathcal{V}_{\mathcal{T}_i} \backslash \{i\}$, and design the transmit signal $X_{t,i}$ to cancel the interference of W_i at Y_r. Finally, it follows from (2.15) that transmitter j is the only transmitter connected to receiver i, and used to deliver W_i. It follows that we can achieve one degree of freedom for each binary variable d_{ij}, and hence, $\eta_c^{\text{avg,zf}}(K,B)$ is lower bounded by the solution of the optimization problem in the theorem statement.

We now describe the converse proof. Consider the optimal zero-forcing scheme achieving $\eta_c^{\text{avg,zf}}(K,B)$. We show that there is a choice of $\{\mathcal{T}_j\}$, $\{d_{ij}\}_{i,j\in[K]}$ satisfying (2.12)-(2.18) such that $\sum_{i\in[K]}\sum_{j\in[K]} d_{ij} \geq \eta_c^{\text{avg,zf}}(K,B)$. Since the considered zero-forcing scheme satisfies the average backhaul load constraint of B, then (2.17) follows by setting $\{\mathcal{T}_j\}$ to the be the set of transmit sets of the considered scheme. Since we achieve zero degrees of freedom for every message whose receiver is inactive, the number of active receivers is at least the achieved degrees of freedom. We further know that since the definition of zero-forcing schemes in Section 2.1.3 ensures the creation of a point-to-point interference-free communication link for each active receiver, then there has to be an optimal zero-forcing scheme achieving $\eta_c^{\text{avg,zf}}(K,B)$, where we achieve one degree of freedom for each message corresponding to an active receiver; we assume that the considered scheme satisfies this property. For each active receiver with an index i, we can hence *assign* a unique active transmitter with an index $j \in \mathcal{T}_i \cap \mathcal{N}_i$, such that $I(W_i; X_{j,i}) > 0$. If transmitter i is assigned to receiver j, then we set $d_{ij} = 1$. Otherwise, we set $d_{ij} = 0$. We then have that (2.12)-(2.16) directly follow. Further, it follows that for any set $\mathcal{S} \subseteq [K]$, $\tilde{\mathcal{V}}_{\mathcal{S}} = \mathcal{V}_{\mathcal{S}}$. We then have that (2.18) follows from Lemma 1, and the converse proof is thus complete.

$\square$

The optimization problem in Theorem 2 is difficult to solve numerically, because we are interested in the asymptotic behavior with large K, and the optimization is over a large number of message assignments, without an explicit bound on the maximum transmit set size constraint M. If a message assigned to n transmitters where $0 \leq n \leq K$, then we have $\binom{K}{n}$ possibilities to choose the transmit set, which is of the order $\mathcal{O}(\min(K^n, K^{K-n}))$. Since we consider a constraint on the average backhaul load and not the maximum transmit size, n can be $\mathcal{O}(K)$ for a particular message. Thus, the computational complexity needed to just consider all message assignments is $\mathcal{O}(K^{\frac{K}{2}})$, i.e., exponential in K.[2]

[2]If we restrict our attention to the irreducible message assignments defined in [9, Section

Hence, instead of trying to solve the optimization problem numerically, we focus on finding upper and lower bounds on the per user DoF.

Interference avoidance

We now restrict ourselves to $M = 1$, and the class of interference avoidance schemes, which is a special case of zero-forcing schemes when $M = 1$, and characterize lower and upper bounds for the maximum achievable per user DoF.

Theorem 3. *The following bounds hold under restriction to interference avoidance schemes for the asymptotic per user DoF of hexagonal cellular networks with no cooperation,*

$$\frac{1}{3} \leq \tau_c^{zf}(M=1) \leq \frac{2}{5}. \tag{2.19}$$

Proof. The proof is available in Section 2.2.3. □

Zero-forcing lower bounds

We now allow for cooperation in the network and show through the results in Theorem 12 and Theorem 5 how a smart choice for assigning messages to transmitters, aided by cooperative transmission, can achieve scalable DoF gains through a zero-forcing coding scheme. For the achievable scheme in Theorem 12, this is done by treating the hexagonal network as interfering locally connected linear networks with connectivity parameter $L = 2$, while the scheme in Theorem 5 considers a division of the network that does not involve linear networks. We note that it follows from Theorem 12 that we can achieve a per user DoF of $\frac{1}{2}$ without requiring an extra load on the backhaul ($B = 1$), which is greater than the $\frac{2}{5}$ upper bound in Theorem 3 for the case without cooperation.

Theorem 4. *Under an integer backhaul load constraint B, the following lower bound holds for the asymptotic per user DoF of the hexagonal cellular*

V-D], then the complexity can be further reduced from $\mathcal{O}\left(K^{\frac{K}{2}}\right)$ to $\mathcal{O}\left(c^{\frac{K}{2}}\right)$, where c is a constant that depends on the number of transmitters connected to a single receiver.

network using zero-forcing schemes:

$$\tau_c^{\text{avg,zf}}(B) \geq \frac{2B}{3B+1}, \forall B \in \mathbf{Z}^+. \tag{2.20}$$

Proof. Consider a division of the network formed by deactivating the nodes in the set Ω_{sq} as shown in Figure 2.3(a). We note that the remaining network consists of non-interfering locally connected subnetworks with connectivity parameter $L = 2$. In each subnetwork, we use the scheme in [9] for $M = 3B$ that considers a division of the subnetwork into non-interfering blocks of $6B + 2$ nodes. The message assignment is shown in Figure 2.3(b) for $B = 1$. This scheme achieves a per user DoF of $\frac{M}{(M+1)}$ with $B = \frac{M}{2}$ in the locally connected linear subnetwork. Since the linear subnetworks only account for $\frac{2}{3}$ of the network, we obtain a per user DoF of $\frac{2B}{3B+1}$ with $B = \frac{M}{3}$ in the entire network.

□

In Theorem 12, $\tau_c^{\text{avg,zf}}(B) \to \frac{2}{3}$ as $B \to \infty$. We now consider achievable schemes which use a different division of the network and show that a per user DoF equal to $\frac{2}{3}$ can be achieved with $B = 4$ with $\tau_c^{\text{avg,zf}}(B) \to \frac{5}{6}$ as $B \to \infty$.

Theorem 5. *Under the average backhaul load constraint B, where $\frac{(5\ell+6)^2}{6\ell+9} \leq B < \frac{(5(\ell+1)+6)^2}{6(\ell+1)+9}$, for some $\ell \in \mathbb{N} \cup \{0\}$, the following lower bound holds for the asymptotic per user DoF of the hexagonal cellular network using zero-forcing schemes:*

$$\tau^{avg,zf}(B) \geq \frac{5\ell+6}{6\ell+9}. \tag{2.21}$$

Proof. The proof is available in Section 2.2.4. □

The achievable values for the per user DoF in Theorems 12 and 5 are compared in Figure 2.4.

Zero-forcing upper bound

Let $\mathcal{K}_{in}$ denote the set of *internal* nodes, i.e., nodes which have five neighbors each, and $\mathcal{K}_{ex}$ denote the set of *external* nodes, i.e., nodes which have less than five neighbors.

We now present the following upper bound on the per user DoF under the average backhaul load constraint B.

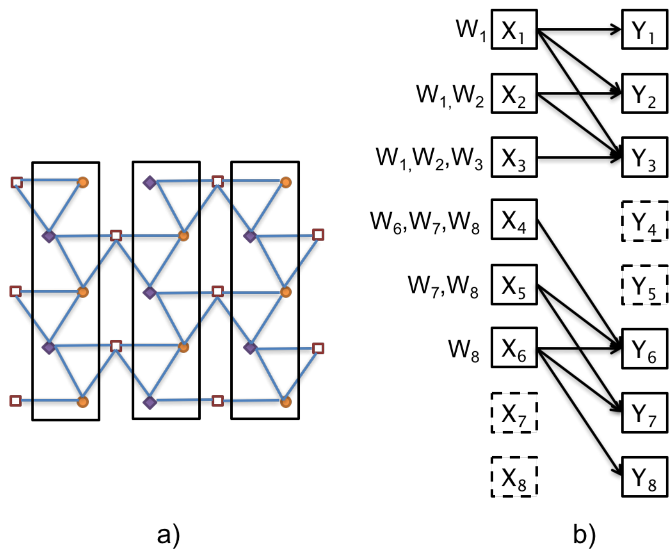


Figure 2.3: (a) Division of cellular network into subnetworks by deactivating nodes in Ω_{sq}, and (b) the message assignment in each subnetwork for $B = 1$. The unshaded nodes in (a) and the transmitters and receivers in the dashed boxes in (b) indicate that they are inactive.

Theorem 6. *Under the average backhaul load constraint B, where $B < 5$, the following upper bound holds for the asymptotic per user DoF under restriction to zero-forcing schemes:*

$$\tau_c^{\text{avg,zf}}(B) \leq \frac{1}{2} + \frac{B}{10}. \tag{2.22}$$

Proof. Consider any message assignment satisfying the average backhaul load constraint of B, and a zero-forcing scheme. Let ρ_j denote the fraction of users whose messages are available at exactly j transmitters, where $0 \leq j \leq K$. We have $\sum_{i=0}^{K} \rho_i = 1$, and from the backhaul load constraint B, we have $\sum_{i=1}^{K} i\rho_i \leq B$. This gives us

$$\sum_{i=2}^{K}(i-1)\rho_i \leq (B-1) + \rho_0. \tag{2.23}$$

We also note that for any given message assignment, the per user DoF is upper bounded by $1 - \rho_0$.

As discussed in the proof of Theorem 2, it follows from our definition of zero-forcing schemes that there is an optimal zero-forcing scheme where we achieve one degree of freedom for each message corresponding to an active

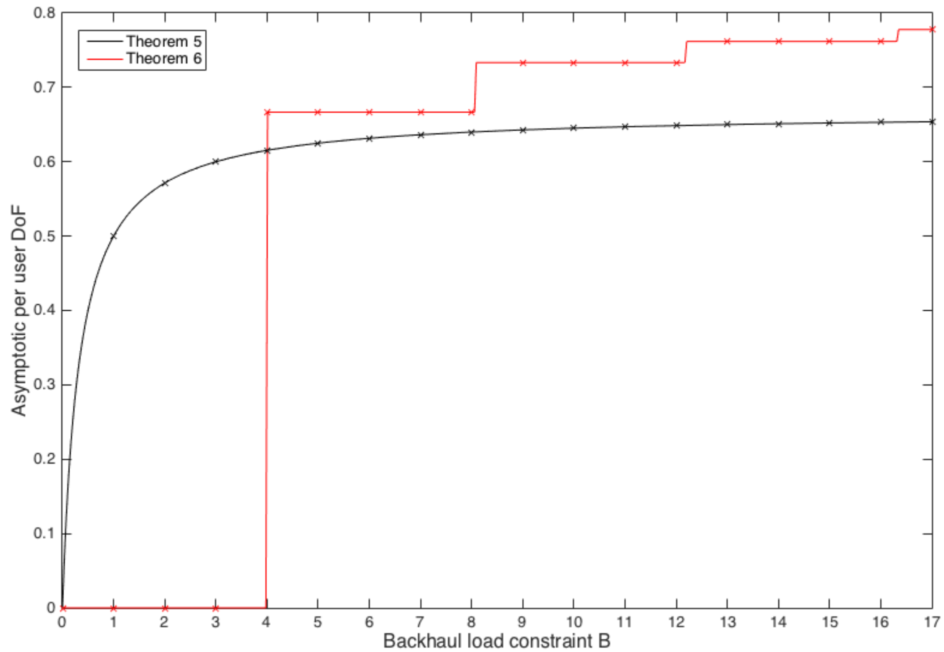


Figure 2.4: Comparison of the lower bounds on the asymptotic per user DoF $\tau_c^{\text{avg,zf}}(B)$.

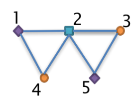


Figure 2.5: Illustration of the neighboring set $\mathcal{N}_j$ for $j = 2$.

receiver. Hence, for each active receiver with an index i, we can assign a unique active transmitter with an index $j \in \mathcal{T}_i \cap \mathcal{N}_i$, such that $I(W_i; X_{j,i}) > 0$.

Consider an active transmitter j that is uniquely assigned to an active receiver i such that $|\mathcal{T}_i| = m$ for some $1 \leq m \leq 4$. In the set $\mathcal{N}_j$ (shown in Figure 2.5), where $\mathcal{N}_j$ denotes the set of five nodes adjacent to node j including node j, from Lemma 1 we have

$$\sum_{k \in \mathcal{N}_j} r_k \leq m, \quad 1 \leq m \leq 4. \tag{2.24}$$

Note that for any transmitter, the number of receivers in the neighboring set is five, and hence the number of active receivers is trivially upper bounded by five. By summing the number of active receivers $\left(\sum_{k \in \mathcal{N}_j} r_k\right)$ in the neighboring set $\mathcal{N}_j$ over all the transmitters $j \in [K]$, we obtain the following:

$$\frac{5\sum_{i\in\mathcal{K}_{in}} r_i + c\sum_{i\in\mathcal{K}_{ex}} r_i}{K} \stackrel{(a)}{\leq} \sum_{i=1}^{4} i\rho_i + \sum_{i=5}^{K} 5\rho_i + 5\rho_0 \tag{2.25}$$

$$\stackrel{(b)}{=} 1 + \sum_{i=2}^{4}(i-1)\rho_i + \sum_{i=5}^{K} 4\rho_i + 4\rho_0 \tag{2.26}$$

$$\stackrel{(c)}{\leq} 1 + (B-1) + \rho_0 + 4\rho_0, \tag{2.27}$$

where $c < 5$ is a positive constant, (a) follows from (2.24), and (b) follows since $\sum_{i=0}^{K} \rho_i = 1$. Finally, (c) follows because (2.23) implies that $\sum_{i=2}^{4}(i-1)\rho_i + \sum_{i=5}^{K} 4\rho_i \leq (B-1) + \rho_0$. Note that each node in the interior of the graph has five neighbors, and hence appears five times on the left hand side of the inequality (a).

We have $|\mathcal{K}_{ex}| = \mathcal{O}(\sqrt{K})$ which gives us $\frac{\sum_{j\in\mathcal{K}_{ex}} r_j}{K} = \frac{\mathcal{O}(\sqrt{K})}{K} \to 0$ as $K \to \infty$. Thus, we have $\tau_c^{\text{avg,zf}}(B) = \lim_{K\to\infty} \frac{\sum_{i\in\mathcal{K}_{in}} r_i}{K} \leq \frac{B+5\rho_0}{5}$. It follows that for any message assignment, the per user DoF is upper bounded by min $\{1-\rho_0, \frac{B}{5}+\rho_0\}$ which gives us $\tau_c^{\text{avg,zf}}(B) \leq \frac{1}{2} + \frac{B}{10}$.

□

We note that the bound in Theorem 6 may not be tight and is useful only for $B < 5$. The comparison between the upper and lower bounds for the per user DoF under zero-forcing schemes is shown in Figure 2.6. We believe that finding a general tight upper bound is difficult, especially for higher values of B, due to the combinatorial search for optimal message assignments as well as the rather complex connectivity structure of hexagonal networks.

2.2.2 Proof of Theorem 1

In this section, before presenting the proof of Theorem 1, we first provide a lemma from [9] for the case of $M = 1$ that serves as a building block for the proof of Theorem 1, and then present a toy network for which we show that the per user DoF is upper bounded by $\frac{1}{2}$ in order to gain some insight into the proof of $\tau_c(M = 1) \leq \frac{1}{2}$.

We present the following lemma for the case of $M = 1$ which gives a relation between the DoF of the message being delivered by a transmitter and the DoF corresponding to the messages of the users connected to that

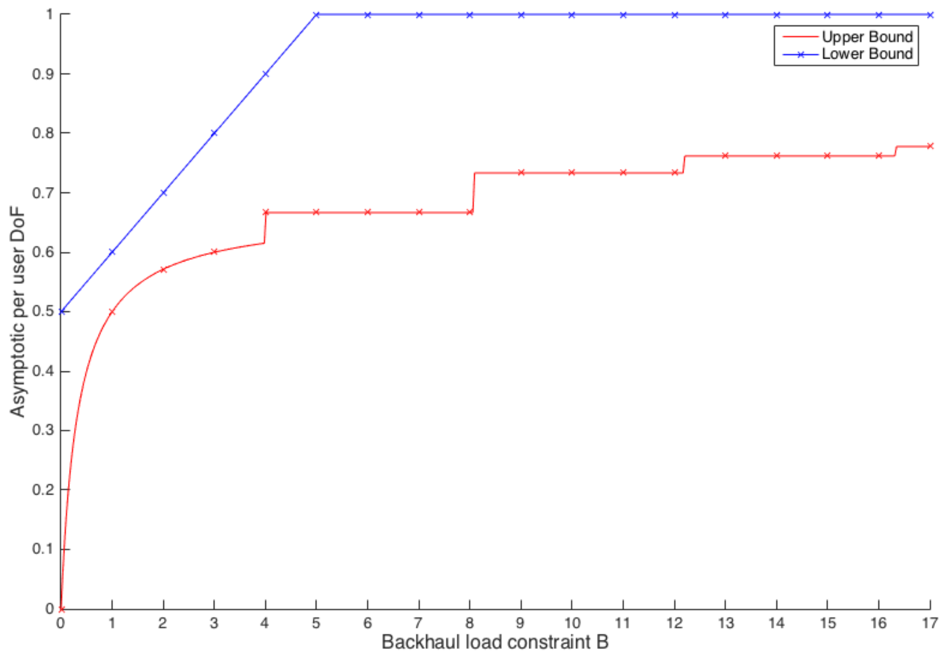


Figure 2.6: Comparison of upper and lower bounds on the asymptotic per user DoF $\tau_c^{\text{avg,zf}}(B)$.

transmitter. Here, $\mathcal{R}_j$ denotes the set of receivers that are connected to transmitter j. The lemma is an extension of the result in [14] which shows that the maximum DoF for a network with two transmitter-receiver pairs is unity.

Lemma 2 ([9, Lemma 5])**.** *If $\mathcal{T}_i = \{X_j\}$, then $d_i + d_k \leq 1$, $\forall k \in \mathcal{R}_j$.*

Each transmitter-receiver pair in the network is referred to as a node. If a and b are two nodes such that they are connected in the connectivity graph, and the transmitter of node a has the message for node b, i.e., $a \in \mathcal{T}_b$, we denote this by $a \to b$.

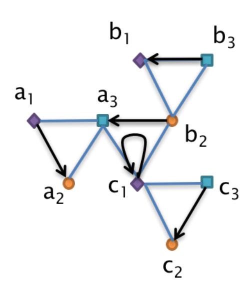


Figure 2.7: An example cellular network with nine transmitter-receiver pairs. The messages of b_3, b_2, a_1, c_3 can be assigned to any transmitter.

Illustrative example

We consider the network and the message assignment shown in Figure 2.7 and show that the per user DoF in the network does not exceed $\frac{1}{2}$ for this particular message assignment. Note that the result holds for any assignment of the messages b_3, b_2, a_1, c_3. Since $a_1 \rightarrow a_2$, we have $d_{a_1} + d_{a_2} \leq 1$ from Lemma 2. Similarly, $b_3 \rightarrow b_1$ and $c_3 \rightarrow c_2$, we have $d_{b_1} + d_{b_3} \leq 1$ and $d_{c_2} + d_{c_3} \leq 1$, respectively from Lemma 2. We now show that $d_{a_3} + d_{b_2} + d_{c_1} \leq \frac{3}{2}$, and hence the per user DoF in this network is upper bounded by $\frac{1}{2}$. Note that $\mathcal{T}_{c_1} = \{c_1\}$ and hence $d_{b_2} + d_{c_1} \leq 1$ and $d_{a_3} + d_{c_1} \leq 1$. We also have $b_2 \rightarrow a_3$ and from Lemma 2, $d_{b_2} + d_{a_3} \leq 1$. Thus $d_{a_3} + d_{b_2} + d_{c_1} \leq \frac{3}{2}$ and the per user DoF in this network is upper bounded by $\frac{1}{2}$.

We now proceed with the proof of Theorem 1.

Consider the division of the network into triangles $\mathcal{D} = \{\Delta(z) : z \in \Omega_{cir}\}$ as shown in Figure 2.8.

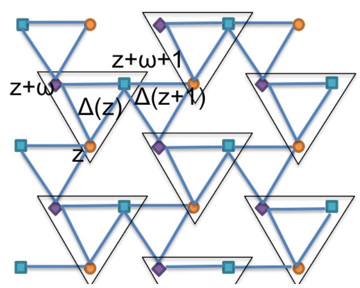


Figure 2.8: Division of the network into triangles.

For any $z \in \Omega_{cir}$, triangle $\Delta(z)$ consists of vertices $z, z+\omega, z+\omega+1$. Note that each triangle contains one vertex from each of the cosets, Ω_{cir}, Ω_{sq} and Ω_{dia}.

We refer to a node as a self-serving node if the message to the receiver corresponding to the node is assigned to its own transmitter. We refer to a node as an outsider node if no message within its triangle is assigned to its transmitter, and also its message is not assigned within its triangle. Let $\mathcal{O}$ denote the set of outsider nodes given by

$$\mathcal{O} = \{i \in \Delta(z) : \Delta(z) \subseteq \mathcal{D}, \mathcal{T}_i \nsubseteq \Delta(z), \mathcal{T}_j \neq \{i\}, \forall j \in \Delta(z)\}.$$

Without loss of generality, we assume that $|\mathcal{T}_j| = 1, \forall j \in [K]$. Note that if the message of a particular receiver is not assigned to any transmitter, then the per user DoF cannot be increased if we assume that the message is assigned to any of the transmitters. We say that a triangle is in state S_i if exactly i of the messages of the triangle are assigned to transmitters within the triangle, $0 \leq i \leq 3$. Let $\mathcal{S}_i$ denote the set of all triangles in state S_i.

$$\mathcal{S}_i = \{\Delta(z) \subseteq \mathcal{D} : \mathbb{1}_{\{\mathcal{T}_z \subseteq \Delta(z)\}} + \mathbb{1}_{\{\mathcal{T}_{z+\omega} \subseteq \Delta(z)\}} + \mathbb{1}_{\{\mathcal{T}_{z+\omega+1} \subseteq \Delta(z)\}} = i\}.$$

Let $\mathcal{SS}_1$ denote the set of all self-serving nodes belonging to triangles in state S_1. More precisely,

$$\mathcal{SS}_1 = \{z : \Delta(z) \in \mathcal{S}_1, \mathcal{T}_z = \{z\}\}.$$

Note that every triangle in state S_0 consists of three outsider nodes, every triangle in state S_1 has at least one outsider node, and a triangle in state S_2 may contain an outsider node.

We also define a middle triangle, as a triangle that is formed by the connected nodes of three different neighboring triangles. Middle triangles are triangles of the form $\{\Delta(z) : z \in \Omega_{dia}\}$. We say that a triangle is associated with a node if the node belongs to the triangle. If $z \in \Omega_{dia}$, the middle triangle associated with vertex z is $\Delta(z)$. If $z \in \Omega_{sq}$, the middle triangle associated with vertex z is $\Delta(z-\omega)$. If $z \in \Omega_{cir}$, the middle triangle associated with vertex z is $\Delta(z-\omega-1)$. For any vertex a, we denote the middle triangle associated with it as M_a. Note that each vertex is associated with exactly one main triangle and one middle triangle. We note that the definition of an outsider node is with respect to the main triangle associated with the node and not the middle triangle associated with it.

Let $\tau_{\mathcal{S}}$ denote the per user DoF for the messages with indices in some set $\mathcal{S}$. We present Algorithm 1, to define a strategy for including nodes in a set $\mathcal{S}$, such that at any stage, the per user DoF of the nodes already included in $\mathcal{S}$ is upper bounded by $\frac{1}{2}$, i.e., $\tau_{\mathcal{S}} \leq \frac{1}{2}$. Note that at the end of the algorithm, all nodes are included in $\mathcal{S}$. To facilitate the understanding of Algorithm 1, we observe the following:

- If $a \in \mathcal{SS}_1$, then a is a self-serving node and since the main triangle T associated with it is in state S_1, the other nodes in the triangle b, c are outsider nodes. We have $d_a + d_b \leq 1$ and $d_a + d_c \leq 1$, according to Lemma 2. Without loss of generality, we include the node with

Algorithm 1

1: Initialize $\mathcal{S} \leftarrow \phi$
2: **while** $\mathcal{S}\mathcal{S}_1 \backslash \mathcal{S} \neq \phi$ **do**
3:     **for** $a \in \mathcal{S}\mathcal{S}_1$ where $a \in \Delta(z)$ for some $z \in \Omega_{cir}$ **do**
4:         $\mathcal{S} \leftarrow \mathcal{S} \cup \{a, j\}$ where $j = \min_{x \in \Delta(z) \backslash \{a\}} \Re(x)$
5:     **end for**
6: **end while**
7: **while** $\mathcal{O} \backslash \mathcal{S} \neq \phi$ **do**
8:     **for** $a \in \mathcal{O} \backslash \mathcal{S}$ where $a \in \Delta(z)$ for some $z \in \Omega_{cir}$ and the associated middle triangle M_a contains nodes b and c apart from a. **do**
9:         **if** $M_a \backslash \mathcal{S}$ contains 3 outsider nodes **then**
10:             $\mathcal{S} \leftarrow \mathcal{S} \cup \{a, b, c\}$
11:         **else if** $M_a \backslash \mathcal{S}$ contains 2 outsider nodes a and j where $j \in \{b, c\}$ **then**
12:             $\mathcal{S} \leftarrow \mathcal{S} \cup \{a, j\}$
13:         **else if** $M_a \backslash \mathcal{S}$ contains a as the only outsider node and message for a is assigned within $M_a \backslash \mathcal{S}$ at $j \in \{b, c\}$, i.e., $j \rightarrow a$ **then**
14:             $\mathcal{S} \leftarrow \mathcal{S} \cup \{a, j\}$
15:         **else if** $M_a \backslash \mathcal{S}$ contains a as the only outsider node and message for a is not assigned within $M_a \backslash \mathcal{S}$ **then**
16:             $\mathcal{S} \leftarrow \mathcal{S} \cup \{a\}$
17:         **end if**
18:     **end for**
19: **end while**
20: **while** $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \backslash S \neq \phi$ **do**
21:     **for** triangle $T \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ **do**
22:         $\mathcal{S} \leftarrow \mathcal{S} \cup T \backslash \mathcal{S}$
23:     **end for**
24: **end while**

minimum real value among the two nodes b, c, and node a in the set $\mathcal{S}$ as in line 4.

- If $M_a \backslash \mathcal{S}$, where M_a is a middle triangle, contains 3 outsider nodes, we include the nodes of that middle triangle a, b, c in the set $\mathcal{S}$ as in line 10. If $M_a \backslash \mathcal{S}$ contains only two outsider nodes a, j, where $j \in \{b, c\}$, we include them in the set $\mathcal{S}$ as in line 12.

  We now show that if nodes are added to the set $\mathcal{S}$ according to line 10 or line 12, then the per user DoF of the nodes included in $\mathcal{S}$ is upper bounded by $\frac{1}{2}$. In any middle triangle with nodes a, b, c, containing at least two outsider nodes, we show that $d_a + d_b \leq 1$, $d_b + d_c \leq 1$,

$d_a + d_c \leq 1$ and hence $d_a + d_b + d_c \leq \frac{3}{2}$. Without loss of generality, let the two outsider nodes be a and b. If the nodes a, b are added according to line 12, it suffices to show that $d_a + d_b \leq 1$ whereas if the nodes a, b, c are added according to line 10, we need to show that $d_a + d_b + d_c \leq \frac{3}{2}$. For node a, we have the following possibilities:

- The message W_a is not available at either b or c. From our assumption, W_a is not available at neighboring nodes outside the triangle. Hence, W_a cannot be transmitted and we have $d_a = 0$.
- The message W_a is available at one vertex in b or c. From lemma 2, we have $d_a + d_c \leq 1$ and $d_a + d_b \leq 1$.

Similarly, for node b, we have $d_b = 0$ if the message W_b is not available at either a or c, or $d_b + d_c \leq 1$ and $d_b + d_a \leq 1$ if the message W_b is available at one vertex in a or c. This gives us $d_a + d_b + d_c \leq \frac{3}{2}$.

Thus, for any middle triangle with nodes a, b, c with at least two outsider nodes, we have $d_a + d_b + d_c \leq \frac{3}{2}$. In addition, we also have $d_a + d_b \leq 1$, $d_b + d_c \leq 1$ and $d_a + d_c \leq 1$ as discussed above. Although for any middle triangle with at least two outsider nodes, the per user DoF is upper bounded by $\frac{1}{2}$, we do not include the third node in the set $\mathcal{S}$ in line 12 in order to simplify the cases considered later.

- Let a be the only outsider node in $M_a \backslash \mathcal{S}$, where M_a is the middle triangle. If its message W_a is available at neighboring node $j \in M_a \backslash \mathcal{S}$ where $j \in \{b, c\}$, i.e., $j \to a$, then we have $d_j + d_a \leq 1$ and include nodes a, j in the set $\mathcal{S}$ as in line 14.

- In the middle triangle M_a, if W_a is not assigned within nodes b, c, we have $d_a = 0$ and we include a in the set $\mathcal{S}$ as in line 16.

  We now consider the case where the message W_a is assigned to a node in the set $M_a \cap \mathcal{S}$ and show that $\tau_{\mathcal{S} \cup \{a\}} \leq \frac{1}{2}$ when we add only the node a in the set $\mathcal{S}$. Suppose $j \to a$ where $j \in \{b, c\}$ but $j \in \mathcal{S}$. We consider the case $j = c$ or $c \to a$ shown in Figure 2.9. So far, we have only added all outsider nodes in a few middle triangles and nodes from self-serving triangles. Hence this is possible only when j was included in $\mathcal{S}$ according to line 4 in the algorithm. Without loss of generality, let j be the self-serving node and m be the outsider node which was included in line 4. We have $d_j + d_a \leq 1$, $d_m + d_a \leq 1$ and we have $d_j + d_m \leq 1$ from before. Note that we have $d_m + d_a \leq 1$ from Theorem 1 since $\mathcal{T}_a = \{j\}$ and $m \in \mathcal{R}_j$. Hence a can be included without any increase in the per user DoF. The same argument holds even if j was the outsider node and m the self-serving node included in line 4. Note that j and m could both contain messages for the only remaining outsider nodes a and k in their respective middle triangles. In that case we see that $d_j + d_a \leq 1$, $d_k + d_m \leq 1$ and $\tau_{\mathcal{S}} \leq \frac{1}{2}$ when k is added later according to line 16.

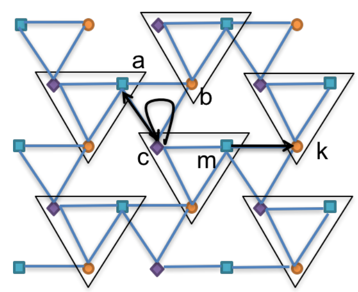


Figure 2.9: Illustration of the case when a is the only outsider node in its middle triangle and its message is available at c where $c \in S$. The node c is a self-serving node and node m has been included according to line 4. The node m contains the message for the only outsider node k in the middle triangle containing m and k.

Consider all triangles in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. If T denotes such a triangle with nodes a, b, c, let t denote the set of nodes in T but not included in $\mathcal{S}$ by line 19. For triangles in $\mathcal{S}_2, \mathcal{S}_3$ with nodes a, b, c, we have $d_a + d_b \leq 1$, $d_b + d_c \leq 1$, $d_a + d_c \leq 1$ and hence $d_a + d_b + d_c \leq \frac{3}{2}$ from Lemma 2. Consider the following cases for any triangle T that has one or more nodes in the set $t = T \backslash \mathcal{S}$:

- The set t contains only one node a. We first find two nodes b, j where $b \to j$ that were previously added to $\mathcal{S}$ according to line 14 and show that $d_j + d_a + d_b \leq \frac{3}{2}$ holds. We then show that nodes b and j do not appear in any other such combination, and hence $\tau_{\mathcal{S}} \leq \frac{1}{2}$ after adding a to $\mathcal{S}$.

  Note that by definition, a triangle in state S_2 or S_3 has at least two messages assigned within the triangle and thus has at least two non-outsider nodes. Hence, if $T \in \mathcal{S}_2 \cup \mathcal{S}_3$, there exists at least one node say b such that b is a non-outsider node and $d_a + d_b \leq 1$. We have the same conclusion if $T \in \mathcal{S}_1$, since all the self-serving nodes and outsider nodes have already been included in $\mathcal{S}$. Hence, it is either the case that $a \to b$ or $b \to a$.

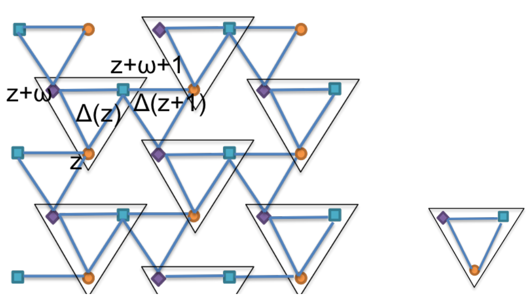
Since b was a non-outsider node that was previously considered, it must have been added according to line 14. Hence, there is an assignment $b \to j$ where j is an outsider node in the middle triangle M_b, $j \in \{a, c\}$ and $d_b + d_j \leq 1$ was considered. We also have $d_a + d_b \leq 1$ and $d_a + d_j \leq 1$ from Lemma 2 since $\mathcal{T}_j = \{b\}$ and $a \in \mathcal{R}_b$. Hence we have $d_j + d_a + d_b \leq \frac{3}{2}$.

Note that neither j nor b is part of any other such combination. This is true for b because all the nodes in its triangle have already been considered. Since $b \to j$ and j has been added to the set $\mathcal{S}$ according to line 14, outsider node j cannot be part of any such combination that does not involve b. Thus, we include $t = \{a\}$ in the set $\mathcal{S}$ as in line 22 while maintaining $\tau_{\mathcal{S}} \leq \frac{1}{2}$.

- The set t contains two nodes say a, b. If $T \in \mathcal{S}_1$, then either $a \to b$ or $b \to a$ and we have $d_a + d_b \leq 1$. If $T \in \mathcal{S}_2 \cup \mathcal{S}_3$, we have $d_a + d_b \leq 1$ and we include $t = \{a, b\}$ in the set S as in line 22.

- The set t contains three nodes a, b, c. This can happen only when $T \in \mathcal{S}_2 \cup \mathcal{S}_3$. In this case, we have $d_a + d_b + d_c \leq \frac{3}{2}$ and we include $t = \{a, b, c\}$ in the set $\mathcal{S}$ as in line 22.

2.2.3 Proof of Theorem 3



(a) (b)

Figure 2.10: Division of network into triangular subnetworks in (a). In (b), we note that by deactivating square and diamond nodes, a per user DoF of $\frac{1}{3}$ is achieved.

Lower Bound Consider the division of the network into triangles $\mathcal{D} = \{\Delta(z) : z \in \Omega_{cir}\}$ as shown in Figure 2.10. For any $z \in \Omega_{cir}$, triangle $\Delta(z)$ consists of vertices $z, z+\omega, z+\omega+1$. By deactivating the nodes $\{z : z \in \Omega_{sqr} \cup \Omega_{dia}\}$, i.e., the square and diamond nodes in each triangle, the network decomposes into $\frac{K}{3}$ isolated nodes $\{z : z \in \Omega_{cir}\}$ that each achieves a DoF of one, thus achieving a per user DoF of $\frac{1}{3}$ in the network.

Upper Bound For each node $j \in \mathcal{K}_{in}$ in the interior of the network, consider the set of neighbors $\mathcal{N}_j$. This results in the block of five nodes as shown in Figure 2.5. For any such j, we show that

$$\sum_{i \in \mathcal{N}_j} \{r_i + t_i\} \leq 4.$$

For any zero-forcing scheme, we first note that among any two adjacent nodes i, j,

$$r_i + t_i + r_j + t_j \leq 2,$$

i.e., among any two adjacent nodes, at least two transmitters or receivers are inactive. This holds because if one of the receivers is active, one transmitter has to be inactive among the nodes $\{i, j\}$ and if one of the transmitters is active, one of the receivers among the nodes $\{i, j\}$ has to be inactive.

We further note that any fully connected triangle in the network is in one of the following states:

State 0 (inactive triangle): All transmitters and receivers in the triangle are inactive.

State 1 (self-serving triangle): Exactly one transmitter in the triangle sends a message to exactly one receiver within the triangle. None of the other transmitters or receivers can be active in this triangle.

State 2 (serving triangle): At least one transmitter in the triangle is activated to serve a receiver in another triangle and there are no active receivers in the considered triangle.

State 3 (served triangle): At least one receiver in the triangle is activated as it is being served by a transmitter in another triangle and there are no active transmitters within the considered triangle.

Without loss of generality we now consider $j = 2$ and the block of five nodes shown in Figure 2.5 and show that $\sum_{i \in \mathcal{N}_2} \{r_i + t_i\} \leq 4$. We show that at least six transmitters or receivers must be inactive. Consider the triangle formed by nodes $\{1,2,4\}$:

- If the triangle is in State 0 then all three transmitters and receivers are inactive and we are done.
- If the triangle is in State 1, then among the three nodes, there is at least one inactive node. Among the remaining adjacent nodes in the triangle at least two of the transmitters or receivers are inactive. Among the nodes $\{3,5\}$, at least two of the transmitters or receivers are inactive. Thus in the block of five nodes, $\sum_{i\in\mathcal{N}_2}\{r_i+t_i\}\leq 4$.
- If the triangle is in State 2, then all three receivers in the triangle are inactive. Suppose all three transmitters in the triangle are active. Then one receiver among nodes $\{3,5\}$ must be receiving message from transmitter 2 and the remaining node among $\{3,5\}$ is inactive. Thus at least six transmitters or receivers are inactive. If on the other hand, at least one transmitter in the triangle is inactive, then we have three inactive receivers and one inactive transmitter within the triangle. Among the nodes $\{3,5\}$, at least two of the transmitters or receivers are inactive. Thus in the block of five nodes, $\sum_{i\in\mathcal{N}_2}\{r_i+t_i\}\leq 4$.
- If the triangle is in State 3, the discussion follows in a similar fashion to the State 2 case with transmitters instead of receivers.

Summing this up over all K users, for some constant $c<5$, we have

$$5\sum_{i\in\mathcal{K}_{in}}\{r_i+t_i\}+c\sum_{j\in\mathcal{K}_{ex}}\{r_j+t_j\}\leq 4K.$$

We have $|\mathcal{K}_{ex}|=\mathcal{O}(\sqrt{K})$ which gives us

$$\frac{\sum_{j\in\mathcal{K}_{ex}}\{r_j+t_j\}}{K}=\frac{\mathcal{O}(\sqrt{K})}{K}\rightarrow 0 \text{ as } K\rightarrow\infty.$$

Thus, we have

$$\tau_c^{\text{avg,zf}}(M=1)\leq \lim_{K\rightarrow\infty}\frac{\sum_{i\in\mathcal{K}_{in}}r_i+t_i}{2K},$$

which gives us per user DoF less than or equal to $\frac{2}{5}$.

2.2.4 Proof of Theorem 5

We first show that under the maximum transmit set size constraint M defined in (2.4), where $5(\ell-1)+6 < M \leq 5\ell+6$, for some $\ell \in \mathbb{N} \cup \{0\}$, a per user DoF of $\frac{M}{6\ell+9}$ can be achieved with an average backhaul load $B = \frac{M^2}{6\ell+9}$ and the proof follows.

For any ℓ, consider the division of the network into blocks of $6\ell+9$ nodes by deactivating the nodes in the set $\mathcal{D}_\ell$, defined as

$$\mathcal{D}_\ell = \left\{ \Delta(z) \bigcup_{m\in[\ell]} \{z - \sqrt{3}m\imath\} : z \in \mathcal{G} \right\},$$

where $\mathcal{G} = \{z : z = (\frac{3}{2}k + 3p) + \imath((2\ell+3)\frac{\sqrt{3}}{2}k), \forall k, p \in \mathbb{Z}\}$, where $\imath = \sqrt{-1}$.

We first prove the result for $1 < M \leq 6$ for which $\ell = 0$ and then extend this scheme to higher values of M which correspond to higher values of ℓ. By deactivating nodes in $\mathcal{D}_0$, the network decomposes into non-interfering blocks containing six nodes each. In the block of six nodes, if M messages are each available at M transmitters, then by the use of simple linear transmit beamforming, we obtain a sum DoF of M thus giving us a per user DoF of $\frac{M}{9}$. Note that for this scheme, the average backhaul load on the network $B = \frac{M^2}{9}$.

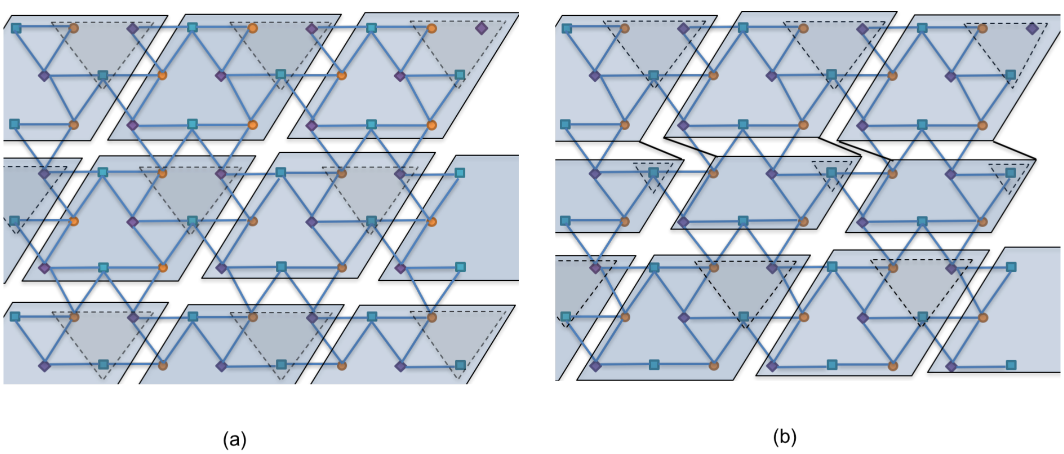


Figure 2.11: Division of cellular network into subnetworks. In (a), $\ell = 0$ and each block has six nodes each. In (b), $\ell = 1$ and each block has a sub-block containing nine nodes and a sub-block containing six nodes below it. The nodes in the triangles denote the deactivated nodes in the network.

For a higher M such that $5(\ell-1)+6 < M \leq 5\ell+6$ with $\ell \geq 1$, consider subnetworks of size $9 + 6\ell$. The case $\ell = 1$ is shown in the Figure 2.11(b). By deactivating the nodes in $\mathcal{D}_\ell$ the network decomposes into non-interfering blocks containing $5\ell + 6$ nodes each. In each non-interfering block, we have a sub-block of six nodes as in the previous case and ℓ sub-blocks containing five nodes each. If in each block, M messages are each available at M transmitters, then by the use of simple linear transmit beamforming, we obtain a sum DoF of M in each block of $6\ell + 9$ nodes. Thus a per user DoF of $\frac{M}{6\ell+9}$ can be attained with an average backhaul load of $\frac{M^2}{6\ell+9}$.

CHAPTER 3

HETEROGENEOUS NETWORKS

In this chapter, we study interference management in the downlink of a heteregeneous network. We consider the downlink of a cellular network as a heterogeneous network consisting of macro basestations (MBs), small cell basestations (SBs), and the mobile terminals (MTs) as illustrated in Figure 3.1. Heterogeneous networks that are built by complementing a macro-cell layer with additional small cells impose new challenges on the backhaul [15]. The best physical location for a small cell often precludes the option of using a wired backhaul. In such cases, deploying a wireless backhaul is both faster and more cost-effective. We consider a point-to-multipoint wireless backhaul between the MBs and the SBs where one MB serves several SBs by sharing its antenna resources. It is assumed that the MBs and the SBs operate on the same frequency band, and that the SBs act as half-duplex analog relays between MBs and MTs.

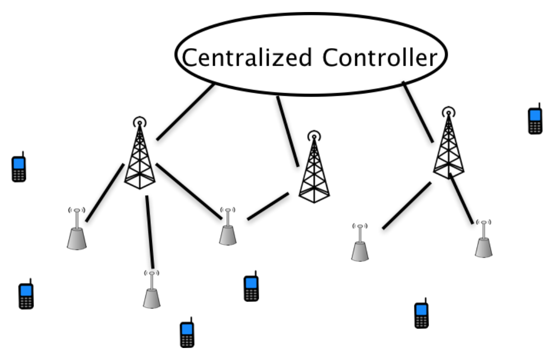


Figure 3.1: A heterogeneous network consisting of a centralized controller, macro basestations, small cell basestations and mobile terminals.

The degrees of freedom (DoF) metric is used to quantify the performance of our proposed schemes. DoF is a high SNR approximation of the capacity of the network that captures the number of interference-free sessions in the network at high SNR. We study the dependence of the DoF in this network on several factors, such as the cluster size S and the number of antennas N

at the MB. We consider a linear network model first, and then study the more practical hexagonal sectored cellular network with and without intra-cell interference in the transmission layer.

The DoF for single-layer locally connected linear networks was discussed in Chapter 2. Cooperation is achieved by making each message available at multiple transmitters and has been shown to significantly increase the achievable DoF.

Prior work on two-layered interference networks includes [16] and [17]. In [16], a two-layered interference network modeled as a $K \times K \times K$ relay channel with each layer as a K-user interference channel with full connectivity is considered. Using aligned-network-diagonalization, the maximum sum-DoF of K is achieved. In [17], the sum-DoF is studied for the special case of $K = 2$, i.e., a $2 \times 2 \times 2$ relay channel, under restriction to linear schemes, and the sum-DoF is shown to be $\frac{4}{3}$. In contrast to these schemes, we consider a broadcast channel in the first layer and local connectivity in both layers. We also restrict ourselves to more practical zero-forcing schemes.

We use insights from previous work on cooperative transmission [18], [9] to characterize the DoF in a two-layered heterogeneous network. Cooperative transmission in the downlink involves sending data to each MT from multiple SBs which share messages through the backhaul connecting the SBs. Joint processing can be used to eliminate interference at the MTs. Typically, such cooperative transmission imposes a high backhaul load since each message needs to be made available at multiple SBs. The backhaul load may increase by a factor of two or three depending on the number of cooperating SBs [9]. Sending multiple messages to different SBs needs additional time-slots since each SB is a half-duplex analog relay. We avoid overloading the backhaul by sending linear combinations of the messages as analog signals to each SB directly so that the corresponding MT can receive its message interference-free. This requires that at each MB, the channel state information (CSI) between SBs in its cluster and the corresponding MTs is known. The availability of CSI is essential for cooperative transmission and reception schemes such as those envisioned in modern cloud RAN networks [19].

3.1 System Model and Notation

We consider the downlink of a heterogeneous cellular network with MBs, SBs and MTs. It is assumed that the MBs do not directly serve the MTs and that the SBs act as half-duplex analog relays between the MBs and the MTs. There are two layers in the network: the wireless *backhaul layer* between MBs and SBs, and the *transmission layer* between SBs and MTs. We assume that the transmissions from the MBs do not cause interference at the MTs. We also assume that the SBs that are actively transmitting do not cause interference at the receiving SBs because transmission in the backhaul layer typically happens at a higher SNR than in the transmission layer and is also more localized.

3.1.1 Backhaul Layer

For the backhaul layer, we consider a point-to-multipoint wireless backhaul where each MB is associated with S SBs. We assume that each MB is equipped with N antennas. Let the channel vector between MB i and SB j at time-slot t be denoted by $\mathbf{h}_{i,j}^B(t)$. Let $\mathbf{x}_i^B(t)$ be the transmitted signal vector from MB i, and let $z_k^B(t)$ denote the additive white Gaussian noise at SB k. The received signal at k-th SB served by MB i is given by

$$y_k^B(t) = (\mathbf{h}_{i,k}^B(t))^T \mathbf{x}_i^B(t) + \sum_{j \neq i} (\mathbf{h}_{j,k}^B(t))^T \mathbf{x}_j^B(t) + z_k^B(t).$$

Local channel state information is assumed to be available at the MBs and SBs. All channel coefficients that are not identically zero are assumed to be drawn independently from a continuous joint distribution.

Linear network

For the linear model, we consider a backhaul layer with connectivity L_B. For any MB i, let $\mathcal{S}_i$ denote the set of S consecutive SBs served by the MB where $\mathcal{S}_i = \{(i-1)S+1, \ldots, (i)S\}$.

$$\mathcal{S}_i(a:b) = \{(i-1)S+a, \ldots, (i-1)S+b\}, \quad \forall i \text{ where } a \leq b \leq S, \text{ and } a:a \equiv a.$$

Each MB i is associated with a set $\mathcal{A}_i$ of $S + L_B$ consecutive SBs illustrated in Figure 3.2(a) where

$$\mathcal{A}_i = \mathcal{S}_{i-1}(\lfloor \frac{L_B}{2} \rfloor : S) \cup \mathcal{S}_i \cup \mathcal{S}_{i+1}(1 : \lceil \frac{L_B}{2} \rceil).$$

Transmission from MB i to any SB in $\mathcal{S}_i$ causes interference at $\lfloor \frac{L_B}{2} \rfloor$ SBs above and at $\lceil \frac{L_B}{2} \rceil$ SBs below the set $\mathcal{S}_i$.

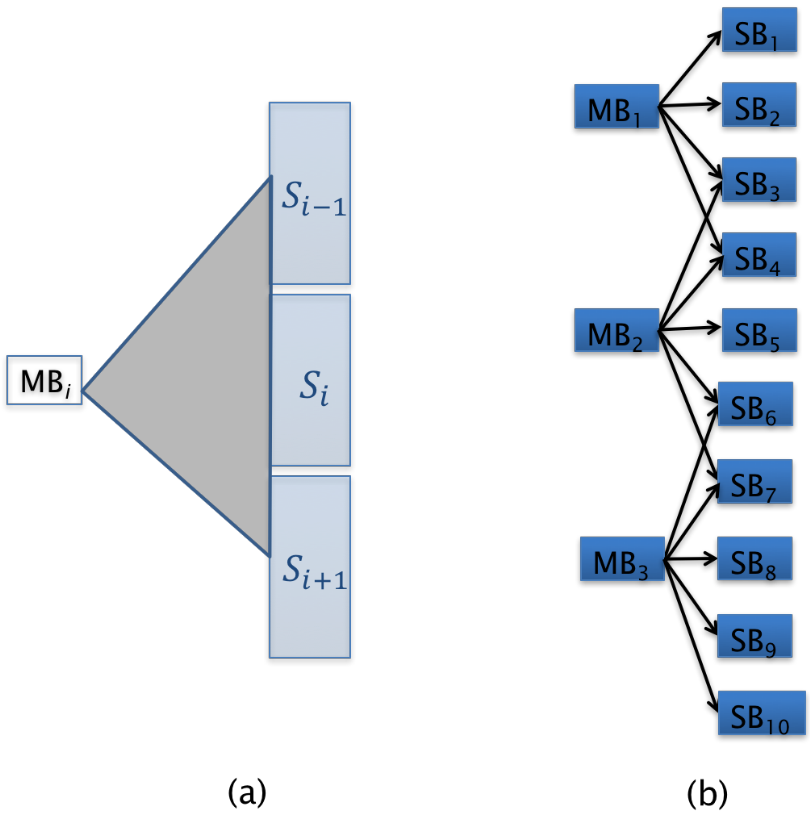


Figure 3.2: In (a), MB i serves the cluster $\mathcal{S}_i$ and causes interference in the preceding and succeeding cluster. In (b), we consider a system with where each MB serves a cluster of three SBs $S = 3$, and $L_B = 2$.

The channel model for backhaul layer is given by $\mathbf{h}^B_{i,j}(t) \neq 0$ iff $j \in \mathcal{A}_i$. The backhaul layer for the linear network is illustrated in Figure 3.2. Let the channel gain matrix corresponding to MB i, $\mathbf{H}^B_i(t) \in \mathbb{C}^{N\times(S+L_B)} = [\mathbf{h}^B_{i,\mathcal{S}_{i-1}(S-\lfloor \frac{L_B}{2} \rfloor)}(t), ..., \mathbf{h}^B_{i,\mathcal{S}_i(S)}(t), \ldots, \mathbf{h}^B_{i,\mathcal{S}_{i+1}(\lceil \frac{L_B}{2} \rceil)}(t)]$ in the backhaul layer where the jth column corresponds to the channel coefficients from MB i to SB j. Let $\mathcal{R}_i(t) \subseteq \mathcal{A}_i$ denote the set of SBs receiving messages from MB i in a particular time-slot t.

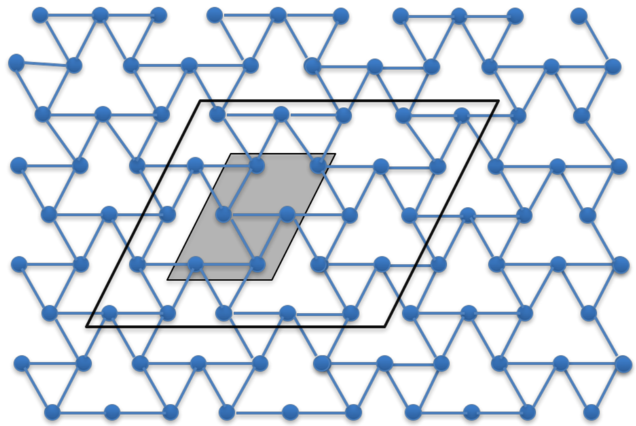


Figure 3.3: $\mathcal{S}_i(2:4, 2:3)$ is denoted by the shaded rectangular box.

Hexagonal network

Each MB (i, j) is associated with a cluster $\mathcal{S}_{(i,j)}$ consisting of S SBs where $\sqrt{S} \in \mathbb{Z}$ where i and j denote the row and column respectively in a two-dimensional grid. The cluster $\mathcal{S}_{(i,j)}$ with S nodes contains $\sqrt{S}$ rows containing $\sqrt{S}$ nodes each, and similarly, $\sqrt{S}$ columns containing $\sqrt{S}$ nodes each. Let $\mathcal{S}_{(i,j)}(a : b, c : d)$, $a \le b, c \le d, 1 \le a, b, c, d \le \sqrt{S}$ denote the set of nodes in the cluster belonging to rows from a to b and columns c to d. Note that here, $a : a \equiv a$. The notation is illustrated in Figure 3.3.

$$\mathcal{S}_{(i,j)}(a : b, c : d) = \{((i-1)\sqrt{S}+a, (j-1)\sqrt{S}+c), \dots, ((i-1)\sqrt{S}+a, (j-1)\sqrt{S}+d)\}$$
$$\cup\{((i-1)\sqrt{S}+a+1, (j-1)\sqrt{S}+c), \dots, ((i-1)\sqrt{S}+a+1, (j-1)\sqrt{S}+d)\}\cup\dots\cup$$
$$\{((i-1)\sqrt{S}+b, (j-1)\sqrt{S}+c), \dots, ((i-1)\sqrt{S}+b, (j-1)\sqrt{S}+d)\}.$$

For a cluster $\mathcal{S}_{(i,j)}$, we define the interior of the cluster as $\mathcal{S}_{(i,j)}(2 : \sqrt{S} - 1, 2 : \sqrt{S} - 1)$, the edges of the cluster as $\mathcal{S}_{(i,j)}(1, 2 : \sqrt{S} - 1), \mathcal{S}_{(i,j)}(2 : \sqrt{S} - 1, 1), \mathcal{S}_{(i,j)}(\sqrt{S}, 2 : \sqrt{S} - 1), \mathcal{S}_{(i,j)}(2 : \sqrt{S} - 1, \sqrt{S})$, and the corner nodes as $\mathcal{S}_{(i,j)}(1, 1), \mathcal{S}_{(i,j)}(1, \sqrt{S}), \mathcal{S}_{(i,j)}(\sqrt{S}, 1), \mathcal{S}_{(i,j)}(\sqrt{S}, \sqrt{S})$. This is illustrated in Figure 3.4.

Each MB (i, j) causes interference at SBs belonging to edges $\mathcal{S}_{(i-1,j)}(\sqrt{S}, :)$, $\mathcal{S}_{(i+1,j)}(1, 1 : \sqrt{S})$, $\mathcal{S}_{(i,j+1)}(1 : \sqrt{S}, 1)$, $\mathcal{S}_{(i,j-1)}(1 : \sqrt{S}, \sqrt{S})$ and at one corner each of the clusters $\mathcal{S}_{(i-1,j-1)}(\sqrt{S}, \sqrt{S})$, $\mathcal{S}_{(i+1,j+1)}(1, 1)$, $\mathcal{S}_{(i+1,j-1)}(\sqrt{S}, 1)$, $\mathcal{S}_{(i-1,j+1)}(1, \sqrt{S})$. This is illustrated in Figure 3.5(a) and 3.5(b). Thus, each MB (i, j) is associated with a set $\mathcal{A}_{(i,j)}$ at which its transmissions can be

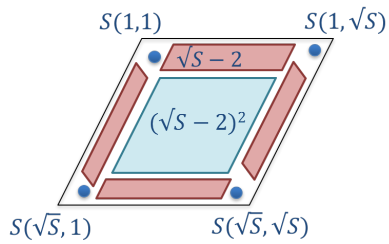


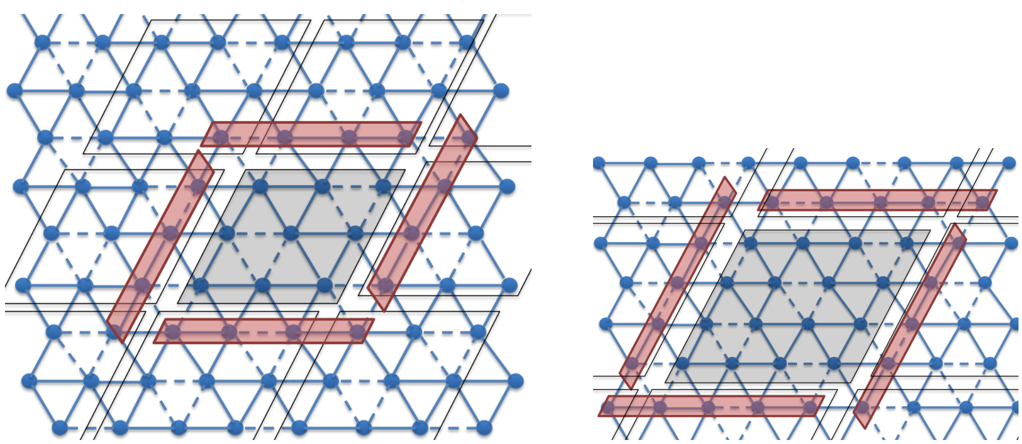
Figure 3.4: The interior of a cluster is the box with $(\sqrt{S}-2)^2$ nodes, the corners are the individual nodes, and the edge nodes are the remaining nodes on the edge outside the interior.

heard where

$$\begin{aligned}\mathcal{A}_{(i,j)} = \mathcal{S}_{(i-1,j)}(\sqrt{S}, 1:\sqrt{S}) \cup \mathcal{S}_{(i+1,j)}(1, 1:\sqrt{S}) \quad \cup \mathcal{S}_{(i,j+1)}(1:\sqrt{S}, 1) \\ \cup \mathcal{S}_{(i,j-1)}(1:\sqrt{S}, \sqrt{S}) \cup \mathcal{S}_{(i,j)} \cup \mathcal{S}_{(i-1,j-1)}(\sqrt{S}, \sqrt{S}) \cup \mathcal{S}_{(i+1,j+1)}(1,1) \\ \cup \mathcal{S}_{(i+1,j-1)}(\sqrt{S}, 1) \cup \mathcal{S}_{(i-1,j+1)}(1, \sqrt{S}).\end{aligned}$$

Let N denote the number of antennas at each MB. The channel vector between MB (i,j) and SB i' is denoted by $\mathbf{h}^B_{(i,j),i'}(t)$. The channel coefficients for the backhaul layer satisfy the condition: $\mathbf{h}^B_{(i,j),i'}(t) \neq 0$ iff $i' \in \mathcal{A}_{(i,j)}$.

Let the channel gain matrix corresponding to MB (i,j), $\mathbf{H}^B_{(i,j)}(t) \in \mathbb{C}^{N\times(S+|\mathcal{A}_{(i,j)}|)}$ in the backhaul layer where the i'th column corresponds to the channel coefficients from MB (i,j) to SB i'.



(a) (b)

Figure 3.5: MB associated with the shaded cluster causes interference at the neighboring $4(\sqrt{S}+1)$ neighbors. In (a), $S=9$ and in (b), $S=16$.

We refer to clusters $\mathcal{S}_{i,j}$ where $i+j$ is even as *shaded* clusters, and the remaining clusters as *white* clusters. This is shown in Figure 3.6.

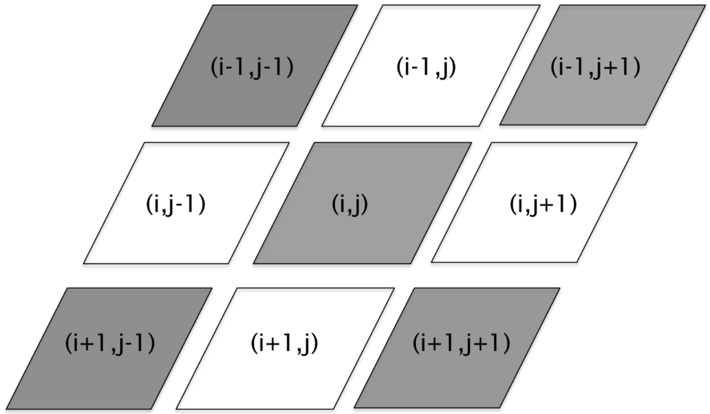


Figure 3.6: Arrangement of shaded clusters and white clusters in the network.

3.1.2 Transmission Layer

Consider the transmission layer with K SBs and K MTs. Let $\mathcal{K}$ denote the set $\{1, ..., K\}$. Each SB and MT is assumed to be equipped with a single antenna. In the transmission layer, the channel gain between SB $j, \forall j \in \mathcal{K}$ and MT $i, \forall i \in \mathcal{K}$ is denoted by h_{ji}^{Tx}. At each MT i, the received signal y_i^{Tx} is given by

$$y_i^{\mathrm{Tx}}(t) = h_{ii}^{\mathrm{Tx}}(t)x_i^{\mathrm{Tx}}(t) + \sum_{j \in \mathcal{I}_i} h_{ji}^{\mathrm{Tx}}(t)x_j^{\mathrm{Tx}}(t) + z_i^{\mathrm{Tx}}(t), \tag{3.1}$$

where t denotes the time-slot, $x_j^{\mathrm{Tx}}(t)$ denotes the signal transmitted by SB j under an average transmit power constraint, $z_i^{\mathrm{Tx}}(t)$ denotes the additive white Gaussian noise at MT i, $h_{ji}^{\mathrm{Tx}}(t)$ denotes the channel gain coefficient from SB j to MT i, and $\mathcal{I}_i^{\mathrm{Tx}}$ denotes the set of interferers at MT i.

Linear network

For the linear network we consider the cellular model presented by Wyner [20] and extended in [9] to a locally connected linear interference network with connectivity parameter L_T. The transmission layer is assumed to be a local L_T-Wyner model with K users. The cells are located on an infinite linear equi-spaced grid, and each transmitter is associated with a single user. Here L_T denotes the number of dominant interferers per user, where each user observes interference from $\lceil \frac{L_T}{2} \rceil$ preceding and $\lfloor \frac{L_T}{2} \rfloor$ succeeding transmitters. The channel coefficients for the L_T-Wyner model are given by

$$h_{ji}^{\mathrm{Tx}}(t) \neq 0 \text{ iff } i \in \{j - \lfloor \frac{L_T}{2} \rfloor, \ldots, j-1, j, j+1, \ldots, j + \lceil \frac{L_T}{2} \rceil\}.$$

The system model is illustrated in Figure 3.7.

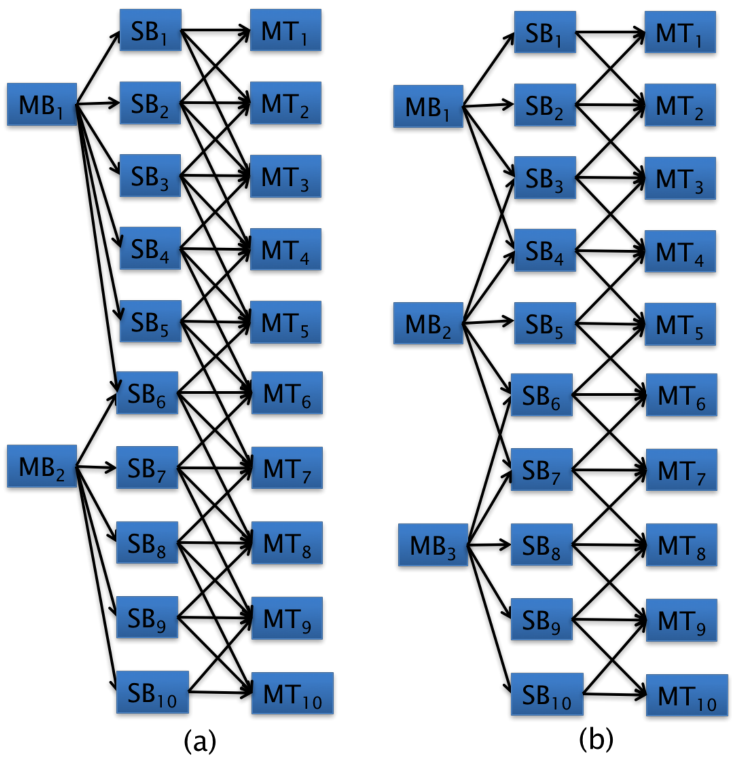
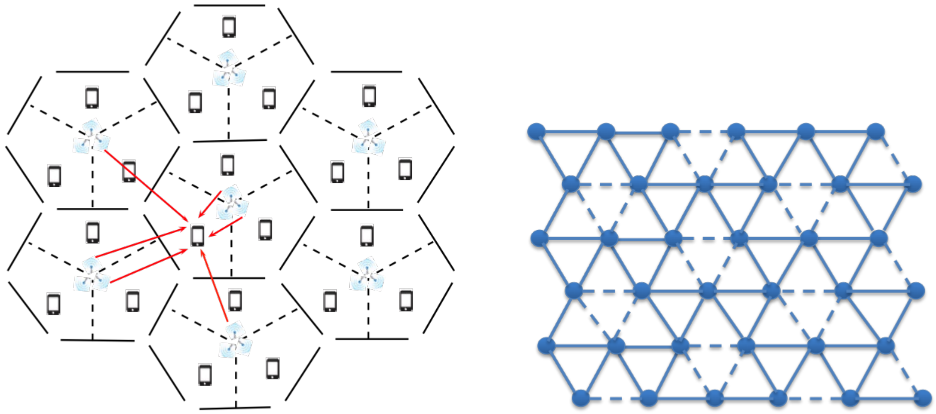


Figure 3.7: Two-layered network with: (a) $S = 5$ and $L_B = 1$ in the backhaul layer, and $L_T = 3$ in the transmission layer; and (b) $S = 3$ and $L_B = 2$ in the backhaul layer, and $L_T = 2$ in the transmission layer.

Hexagonal network

For the hexagonal network, we consider a sectored K-user network with three sectors per cell as shown in Figure 3.8(a). A local interference model is assumed, where the interference at each receiver is only due to the basestations in the neighboring sectors. We consider two models for the transmission layer. In the first model, we assume that sectors belonging to the same cell



(a) (b)

Figure 3.8: (a) Cellular network and (b) interference graph. The dotted lines in (b) represent interference between sectors belonging to the same cell.

do not interfere with each other. In the second model, we assume that sectors belonging to the same cell do interfere with each other.

Interference graph

The cellular model is represented by an undirected interference graph $G(V, E)$ shown in Figure 3.8(b) where each vertex $u \in V$ corresponds to a transmitter-receiver pair. For any node a, the transmitter, receiver and intended message corresponding to the node are denoted by T_a, R_a and W_a, respectively. An edge $e \in E$ between two vertices $u, v \in V$ corresponds to interference between the transmit-receiver pairs, i.e., the transmitter at u causes interference at the receiver at v, and vice-versa. The dotted lines denote interference between sectors that belong to the same cell. Depending on the model we consider for the transmission layer, the dotted lines may or may not be present in the interference graph.

3.1.3 Capacity and Degrees of Freedom

Let P be the average transmit power constraint at each SB and the transmit power per antenna at each MB. Let $\mathcal{W}_i$ denote the alphabet for W_i, where W_i denotes the message for MT i. The rates $R_i(P) = \frac{\log|\mathcal{W}_i|}{n}$ are achievable iff the error probabilities of all messages can simultaneously be made arbitrarily small for large n, using an interference management scheme. The degree of freedom (DoF) $d_i, \forall i \in \mathcal{K}$ is defined as

$$d_i = \lim_{P \to \infty} \frac{R_i(P)}{\log P}. \tag{3.2}$$

DoF corresponds to the number of interference-free sessions that can be accommodated in a multi-user channel. The maximum achievable sum DoF $\eta(K)$ in a channel with K users (MTs) is defined as

$$\eta(K) = \max_{\mathcal{D}} \sum_{i \in \mathcal{K}} d_i,$$

where $\mathcal{D}$ denotes the closure of the set of all achievable DoF tuples, and the per user DoF τ_K is defined as

$$\tau_K = \frac{\eta(K)}{K} \tag{3.3}$$

with $\tau_\infty = \lim_{K \to \infty} \tau_K$.

3.2 Linear Network

We consider the linear network and show that the optimal puDoF can be achieved using only zero-forcing schemes. We show that for lower connectivity in the transmission layer, i.e., $L_T = \{1, 2\}$, the optimal puDoF can be achieved without any cooperation in the network but as the transmission layer connectivity L_T increases, cooperation in the network becomes crucial in order to achieve the optimal puDoF.

We first present an upper bound on the per user DoF for any general heterogeneous network.

Theorem 7. *The following upper bound holds for the asymptotic per user DoF τ_∞, for any cellular network model for the transmission layer and when each MB has N antennas and a cluster size of S,*

$$\tau_\infty \leq \min\left(\frac{N}{S}, \frac{1}{2}\right). \tag{3.4}$$

Proof. Consider any SB i. For every message SB i sends, one time-slot is required in receiving that message in the backhaul layer due to the half-duplex nature of the SB. After T time-slots, the maximum number of messages transmitted by each SB i is $\frac{T}{2}$ and that received by all MTs is $\frac{K(T-1)}{2}$. Thus, for any scheme, the puDoF is given by

$$\frac{\text{No. of messages received interference-free}}{KT} \leq \frac{K(T-1)}{2KT} < \frac{1}{2}.$$

The number of macro basestations is $\lceil \frac{K}{S} \rceil$. After T time-slots, the maximum number of messages that can be received by the K SBs is $\lceil \frac{K}{S} \rceil TN$ messages. Hence the maximum number of messages that can be received by the MTs is $\lceil \frac{K}{S} \rceil (T-1)N$. Thus, for any scheme, the puDoF is given by

$$\tau_K \leq \frac{\lceil \frac{K}{S} \rceil (T-1)N}{KT} \leq (\frac{1}{S} + \frac{1}{K})N(1 - \frac{1}{T}).$$

The upper bound approaches $\frac{N}{S}$ when T and K become large. We have $\tau_K \leq \min\{(\frac{1}{S} + \frac{1}{K})N, \frac{1}{2}\}$ and thus $\tau_\infty \leq \min(\frac{1}{2}, \frac{N}{S})$. □

3.2.1 DoF Analysis for $L_T = \{1, 2\}$

We now consider the case where the connectivity in the transmission layer $L_T \leq 2$ while $L_B = 1$. We present lower bounds on the achievable puDoF which hold for general L_T and L_B. Similar achievability schemes can be used for higher values of L_T and L_B.

Note that at any MB i', $N_1 + N_2$ antennas are sufficient in order to send messages to N_1 SBs and to null the interference at N_2 SBs. Let $\mathcal{R}_{i'}$ denote the set of SBs that are receiving the messages and $\mathcal{Z}_{i'}$ denote the set of SBs at which interference is being zero-forced. Let $|\mathcal{R}_{i'}| = N_1$ and$|\mathcal{Z}_{i'}| = N_2$. Note that $\mathcal{R}_{i'}, \mathcal{Z}_{i'} \subseteq \mathcal{A}_{i'}$. Let $\mathbf{X} \in \mathbb{C}^{N_1+N_2}$ denote the transmitted signal vector at MB i', $\mathbf{H}$ denote $[\mathbf{H}^B_{i',\mathcal{R}_{i'}}, \mathbf{h}^B_{i',\mathcal{Z}_{i'}}]$, and $\mathbf{W} \in \mathbb{C}^{N_1+N_2}$ denote the vector containing the intended messages to $\mathcal{R}_{i'}$ appended with zero at the end. Then we have $\mathbf{H}\mathbf{X}^T = \mathbf{W}^T$. From our assumptions, $\mathbf{H}$ is full rank almost surely and the solution $\mathbf{X} = (\mathbf{H}\mathbf{H}^*)^{-1}\mathbf{H}\mathbf{W}$ is obtained.

Theorem 8. *The following lower bound holds for the asymptotic puDoF τ_∞, for a linear heterogeneous network when the backhaul layer connectivity*

$L_B = 1$ *and the transmission layer connectivity* $L_T \in \{1, 2\}$,

$$\tau_\infty \geq \begin{cases} \frac{N}{S} & \text{for } N < \frac{S}{2} \\ \frac{1}{2}(1 - \frac{1}{S}) & \text{for } N = \frac{S}{2} \text{ for } S \text{ even.} \\ \frac{1}{2} & \text{for } N > \frac{S}{2} \end{cases} \tag{3.5}$$

Proof. In the transmission layer for the L_T-Wyner model with $L_T \in \{1, 2\}$, by deactivating alternate transceiver pairs, the remaining messages can be sent interference-free as shown in Figure 3.9. Thus, a puDoF of $\frac{1}{2}$ is achieved if the corresponding messages are available at the active SBs.

Case 1: $N > \frac{S}{2}$.

A) When S is odd, our achievable scheme uses only $\frac{S+1}{2}$ antennas at an MB. Consider the following message assignment for each time-slot t where t is odd.

$$\mathcal{R}_i(t) = \begin{matrix} \{\mathcal{S}_i(1), \mathcal{S}_i(3), ..., \mathcal{S}_i(S)\} & \text{for } i \text{ odd} \\ \{\mathcal{S}_i(2), \mathcal{S}_i(4), ..., \mathcal{S}_i(S-1)\} & \text{for } i \text{ even.} \end{matrix}$$

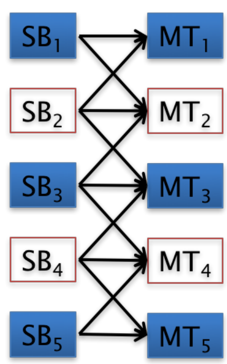


Figure 3.9: Scheme achieving puDoF of $\frac{1}{2}$ in the transmission layer with $L_T = 2$. The unshaded boxes indicate deactivated transceivers.

When i is even, SB $\mathcal{S}_i(1)$ is not active in this time-slot. Only when i is odd, does $\mathcal{S}_i(1)$ observe interference from the transmissions of MB $i-1$. MB $i-1$ needs $\frac{S-1}{2}$ antennas for sending messages and one antenna for nulling the interference at SB $\mathcal{S}_i(1)$. Thus at the end of each odd time-slot, messages are available at alternate SBs, and a puDoF of $\frac{1}{2}$ is achieved. The assignment is reversed when t is even, and the achievability follows similarly.

B) When S is even, our achievable scheme uses $\frac{S}{2} + 1$ antennas at each MB. In odd and even time-slots, only the odd and even numbered SBs are served, respectively. This is possible as the cluster of any MB i contains $\frac{S}{2}$ SBs with odd indices and $\frac{S}{2}$ SBs with even indices. Only in time-slots t, where t is odd, does the first SB in each active cluster observe interference. Each MB uses $\frac{S}{2}$ antennas to send messages and has an additional antenna to null interference at the first SB in the next cluster. Thus, in each time-slot, messages are available at alternate SBs and a puDoF of $\frac{1}{2}$ is achieved.

Case 2: $N < \frac{S}{2}$. In this case, $S \geq 2N+2$ or $S \geq 2N+1$ for even and odd indices, respectively. Hence in each cluster, two disjoint sets of N SBs are served in consecutive time-slots while the first SB of the cluster is inactive. Consider the following message assignment for each time-slot t when t is odd:

$$\mathcal{R}_i(t) = \begin{array}{ll} \{\mathcal{S}_i(3), \mathcal{S}_i(5), ..., \mathcal{S}_i(2N+1)\} & \text{for } i \text{ odd} \\ \{\mathcal{S}_i(2), \mathcal{S}_i(4), ..., \mathcal{S}_i(2N)\} & \text{for } i \text{ even.} \end{array}$$

This assignment is reversed when t is even. The first SB in each cluster is not served at all and hence there is no interference in the backhaul layer. In each time-slot, N messages among every S users are sent interference-free, achieving a puDoF of $\frac{N}{S}$.

Case 3: $N = \frac{S}{2}$. This case arises only when S is even. For an even time-slot t, let the even numbered SBs be served. Only the first SB in each cluster sees interference, and hence there is no interference in the backhaul layer in this time-slot. When t is odd, all the odd numbered indices $(\frac{S}{2}-1)$ except for the first ones in each cluster are served. In the transmission layer, these messages are sent interference-free and a puDoF of $\frac{1}{2}(\frac{1}{2}+\frac{S/2-1}{S})$ is achieved. □

From Theorem 8 it follows that the upper bound in (3.4), i.e., the maximum puDoF can be achieved by simple interference avoidance schemes except for the case $N = \frac{S}{2}$ when $L_T \in \{1, 2\}$. The achievable schemes are illustrated in Figure 3.10. Even for a general L_B, by employing a sufficiently large number of antennas $N \geq \lceil \frac{S}{2} \rceil + L_B$ at the MBs, the interference in backhaul layer can be eliminated and a puDoF of $\frac{1}{2}$ can be achieved.

3.2.2 Achievable Schemes for General L_T

The optimal puDoF for a given number of antennas cannot be achieved for higher values of L_T using only interference avoidance schemes without the use of cooperation. For example, when $L_T = 3$, with restriction to only ZF

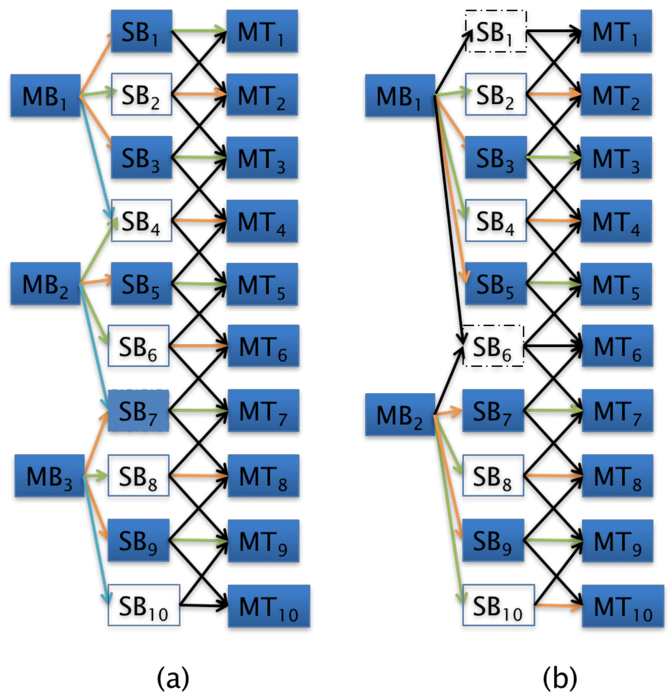


Figure 3.10: Achievable schemes for the network with $L_B = 1$ and $L_T = 2$: (a) puDoF of $\frac{1}{2}$ with $S = 3$ and $N = 2$; and (b) puDoF of $\frac{2}{5}$ with $S = 5$ and $N = 2$. The shaded and unshaded SBs receive messages in alternate time-slots and the dashed SBs do not receive messages.

schemes without cooperation at the SBs, we have $\tau_\infty \leq \frac{2}{5}$ in the transmission layer even for a large N (see e.g., [9]). We consider cooperation among the SBs and show that the optimal puDoF can be achieved for $L_T \in \{3, 4\}$ using only interference avoidance schemes. For cooperation, multiple messages need to be available at SBs for transmission in a particular time-slot, which requires multiple time-slots for transmission by the MBs in the backhaul layer and leads to ineffective use of resources. The SBs use the knowledge of messages available only for zero-forcing, and, thus, it suffices to have a linear combination of messages at the SBs. Transmission of a linear combination of messages to the SBs requires only one time-slot in the backhaul layer. However, this would require that at each MB, the channel between the SBs in its cluster and the corresponding MTs is known. The requirement of a large amount of CSI to be present at each MB is justified if the coherence time is large enough.

Remark 1. *In the Wyner L_T model, if groups of A consecutive SB-MT pairs are separated by F deactivated pairs where $F \geq \lceil \frac{L_T}{2} \rceil$, then there is* *no interference between the groups. If all A messages are sent such that the interference at each MT is zero-forced, a puDoF of $\frac{A}{F+A}$ is achieved if the messages are available at the corresponding SBs.*

Theorem 9. *The following lower bound holds for the asymptotic puDoF τ_∞, for a linear heterogeneous network when the backhaul layer connectivity $L_B = 1$ and the transmission layer connectivity L_T and the cluster size S are such that $\lfloor \frac{S}{2} \rfloor \geq \lceil \frac{L_T}{2} \rceil$*

$$\tau_\infty \geq \begin{cases} \frac{N}{S} & \text{for } N < \frac{S}{2} \\ \frac{1}{2}(1 - \frac{1}{S}) & \text{for } N = \frac{S}{2} \text{ for } S \text{ even.} \\ \frac{1}{2} & \text{for } N > \frac{S}{2} \end{cases} \tag{3.6}$$

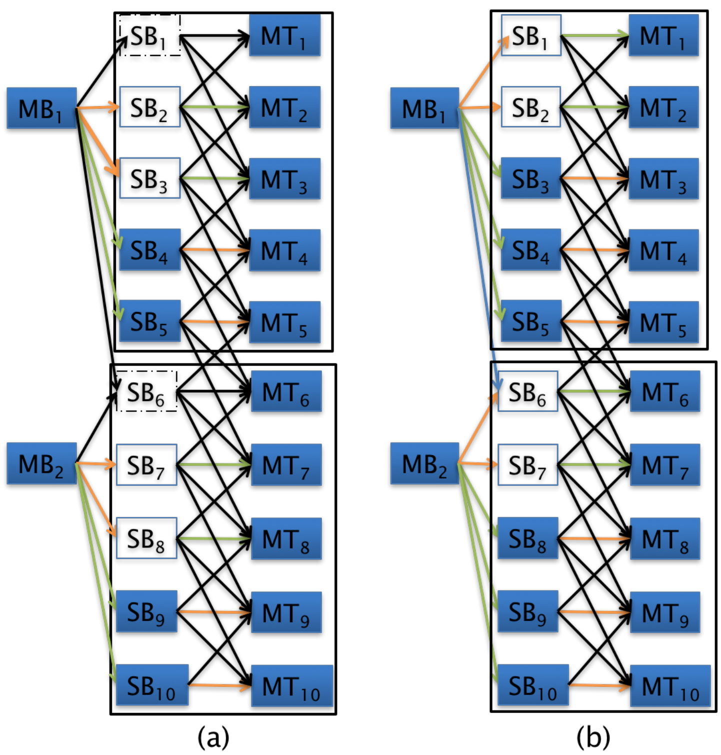


Figure 3.11: Achievable schemes for the network with $L_B = 1$, $L_T = 3$ and $S = 5$. In (a), $N = 2$, $N < \frac{S}{2}$, and puDoF of $\frac{2}{5}$ is achieved. In (b), $N = 3$, $N > \frac{S}{2}$, and puDoF of $\frac{1}{2}$ is achieved. The shaded and unshaded SBs receive messages in alternate time-slots, and the dashed SBs do not receive messages.

Proof. 1. $N > \frac{S}{2}$ is equivalent to $N \geq \lfloor \frac{S}{2} \rfloor + 1$. For all i, let

$$\mathcal{R}_i(t) = \begin{array}{ll} \mathcal{S}_i(1 : \lfloor \frac{S}{2} \rfloor) & \text{for } t \text{ odd} \\ \mathcal{S}_i(\lfloor \frac{S}{2} \rfloor + 1 : S) & \text{for } t \text{ even.} \end{array}$$

In even and odd time-slots, $\lfloor \frac{S}{2} \rfloor + 1$ and $\lfloor \frac{S}{2} \rfloor$ antennas, respectively, at each MB i are used to send linear combinations to the SBs, and in an odd time-slot one antenna is used to ZF interference at $\mathcal{S}_{i+1}(1)$. From Remark 1, it follows that the puDoF is $\frac{1}{2}$.

2. $N < \frac{S}{2}$ is equivalent to $N \leq \lceil \frac{S}{2} \rceil - 1$. For all i, let

$$\mathcal{R}_i(t) = \begin{array}{ll} \mathcal{S}_i(2 : N + 1) & \text{for } t \text{ odd} \\ \mathcal{S}_i(\lceil \frac{S}{2} \rceil + 1 : \lceil \frac{S}{2} \rceil + 1 + N) & \text{for } t \text{ even.} \end{array}$$

In each time-slot, N antennas at each MB i send linear combinations to the SBs. The first SB in each cluster is always inactive. Each group of N SBs is separated by $S - N$ SBs and hence from Remark 1, the puDoF is $\frac{N}{S}$.

3. $N = \frac{S}{2}$. This case arises when S is even. For all i,

$$\mathcal{R}_i(t) = \begin{array}{ll} \mathcal{S}_i(2 : \frac{S}{2}) & \text{for } t \text{ odd} \\ \mathcal{S}_i(\frac{S}{2} + 1 : S) & \text{for } t \text{ even.} \end{array}$$

In the odd and even time-slots, $N - 1$ and N antennas, respectively, at each MB i are used to send linear combinations to the SBs. Hence we achieve a puDoF of $\frac{N}{S}$ and $N - \frac{1}{S}$ in consecutive time-slots, giving an average puDoF of $\frac{2N-1}{2S}$.

□

The achievable schemes are illustrated in Figure 3.11. We note that for a general L_B, a puDoF of $\frac{1}{2}$ can be achieved by employing a sufficient number of antennas ($N \geq \lfloor \frac{S}{2} \rfloor + L_B$).

3.3 Hexagonal Cellular Network

The two-layered linear network is a much simpler interference network compared to the two-layered hexagonal network. We use these insights from the simple linear network and extend the results to the more realistic hexagonal network that has a complicated interference pattern.

We now consider the hexagonal sectored cellular network, with and without intra-cell interference. Unless mentioned otherwise, the results hold for both the network models, i.e., with and without intra-cell interference.

We now discuss achievable schemes for the network for different number of antennas N at each MB. We use the idea of zero-forcing in the backhaul layer similar to the schemes in Section 3.2.

We note that the achievable schemes do not require any cooperation between the MBs but do require that linear combinations be sent by the MBs to SBs to zero-force the interference at the MTs.

Theorem 10. *The following lower bound holds for the asymptotic puDoF τ_∞, for the hexagonal heterogeneous cellular network with or without intra-cell interference,*

$$\tau_\infty \geq \begin{cases} \frac{N}{S} & \textit{for } N < \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil, \\ \frac{N+\lfloor(\sqrt{S}-2)^2/2\rfloor}{2S} & \textit{for } \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil \leq N \leq \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil + 4(\sqrt{S}-2), \\ \frac{1}{2} - \frac{1}{S} & \textit{for } N \geq \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil + 4(\sqrt{S}-2) + 6. \end{cases}$$

Proof. We refer to clusters $\mathcal{S}_{(i,j)}$, where $i+j$ is even as shaded clusters, and the rest as white clusters, as discussed in Section 3.1.1 and shown in Figure 3.6. Interior, edge and corner nodes were introduced in Section 3.1.1 and shown in Figure 3.4.

1. $N < \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil$
   Backhaul Layer: For $N < \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil$, we show that in each time-slot, N messages are sent to SBs interference-free in the backhaul layer. In each time-slot we send linear combinations of messages to N SBs in the interior $(\sqrt{S}-2)^2$ SBs of each cluster. Since $N < \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil$, in each time-slot we can find a new set of N SBs in the interior of the cluster which did not receive a message in the previous time-slot. In the backhaul layer, the outer nodes in each cluster observe interference

from transmissions of neighboring MBs. Since there are no transmissions to the outer nodes in each cluster, there is no interference in the backhaul layer in this case.

Transmission Layer: In the transmission layer, we need to show that N messages can be sent interference-free in each cluster. There is no interference in the transmission layer across clusters because the active SBs are within the interior of each cluster. Within each cluster, linear combinations of messages are sent by the MB in a way to ZF interference and thus the messages are sent interference-free. Hence in each time-slot, N MTs receive their messages interference-free in each cluster consisting of S MTs, achieving a per user DoF of $\frac{N}{S}$.

2. $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil \leq N \leq \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil + 4(\sqrt{S}-2)$.
Note that the per user DoF in this case is $\frac{N}{S}$ if $\sqrt{S}$ is even and $\frac{N-1}{S}$ otherwise.
Backhaul Layer:
    - Shaded clusters: In odd time-slots, $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ SBs in the interior of the shaded clusters receive messages and $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ antennas zero-force the interference at the edge nodes in neighboring white clusters. In even time-slots, $\lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ SBs corresponding to the interior of the shaded clusters receive messages and $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ of the edge nodes in the exterior receive messages. Note that the exterior SBs do not observe interference because the MBs corresponding to white clusters zero-force the interference at these SBs in even time-slots.
    - White clusters: In even time-slots, $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ SBs corresponding to the interior of the white clusters receive messages and $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ antennas zero-force the interference at the edge nodes in neighboring clusters. In odd time-slots, $\lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ SBs corresponding to the interior of the white clusters receive messages and $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ of the edge nodes in the exterior receive messages. Note that the exterior SBs do not observe interference in the odd time-slots because the MBs corresponding to neighboring shaded clusters ZF the interference at these SBs.

Transmission Layer: The interior nodes of a cluster do not observe interference in the transmission layer. In even time-slots, the $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ edge MTs corresponding to the shaded clusters receive messages, and in odd time-slots, the $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ edge MTs corresponding to the white clusters receive messages from their respective SBs. In each time-slot, we notice that there is no interference at the MTs. This is because edge SBs in shaded clusters cause interference only at MTs belonging to neighboring white clusters, and vice-versa. Within the cluster, linear combinations of messages are sent by the MBs to zero force the interference, and hence the messages are received interference-free at the MTs. Thus, over two time-slots $N + \lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ messages are sent interference-free.

3. $N = \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil + 4(\sqrt{S} - 2) + 6$
   Note that in this scheme, over two consecutive time-slots, the message of the MT $\mathcal{S}_{(i,j)}(\sqrt{S}, \sqrt{S})$ is sent interference-free for all clusters, and in the next two time-slots, the message of the MT $\mathcal{S}_{(i,j)}(1, 1)$ is sent interference-free for all clusters. We alternate between sending the messages of MT $\mathcal{S}_{(i,j)}(\sqrt{S}, \sqrt{S})$ over two consecutive time-slots and MT $\mathcal{S}_{(i,j)}(1, 1)$ over the next two consecutive time-slots.
   Backhaul Layer:

   - Shaded clusters: In odd time-slots, $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ SBs in the interior of the shaded clusters receive messages and $4(\sqrt{S} - 2) + 6$ antennas zero-force interference at the edge and corner nodes in neighboring white clusters. In even time-slots, $\lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ SBs corresponding to the interior and $4(\sqrt{S}-2)$ edge nodes and three corner nodes in the exterior of the shaded clusters receive messages. The interference at the three corner nodes of the neighboring shaded clusters is zero-forced by three additional antennas. Note that the exterior SBs do not observe interference because the MBs corresponding to white clusters zero-force interference at these SBs.
   - White clusters: In even time-slots, $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ SBs corresponding to the interior of the white clusters receive messages, $4(\sqrt{S} - 2)$ antennas zero-force interference at the edge nodes, and six antennas zero-force interference at the corner nodes in neighboring shaded clusters. In odd time-slots, $\lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ SBs corresponding to the interior, $4(\sqrt{S} - 2)$ edge nodes, and three corner nodes in

the exterior of the white clusters receive messages. The interference at the three corner nodes of the neighboring white clusters is zero-forced by three additional antennas. Note that the exterior SBs do not observe interference in the odd time-slots because the MBs corresponding to neighboring shaded clusters zero-force the interference at these SBs.

Transmission Layer: The interior nodes of a cluster do not observe interference in the transmission layer. In even time-slots, the $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ edge MTs corresponding to the shaded clusters receive messages, and in odd time-slots, the $N - \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ edge MTs corresponding to the white clusters receive messages from their respective SBs. In every shaded (or white) cluster $\mathcal{S}_{(i,j)}$, there are two corner nodes $\mathcal{S}_{(i,j)}(1, \sqrt{S}), \mathcal{S}_{(i,j)}(\sqrt{S}, 1)$ that do not cause interference in the transmission layer at the corner nodes in neighboring shaded (or white) clusters $\mathcal{S}_{(i-1,j+1)}(\sqrt{S}, 1)$, $\mathcal{S}_{(i+1,j-1)}(1, \sqrt{S})$. In every shaded (or white) cluster $\mathcal{S}_{(i,j)}$, the remaining two corner nodes $\mathcal{S}_{(i,j)}(1, 1)$, $\mathcal{S}_{(i,j)}(\sqrt{S}, \sqrt{S})$ cause interference at their respective corner nodes $\mathcal{S}_{(i-1,j-1)}(\sqrt{S}, \sqrt{S})$, $\mathcal{S}_{(i+1,j+1)}(1, 1)$ in neighboring shaded (or white) clusters. The messages to the corner nodes that do not cause interference at neighboring clusters are sent interference-free. Among the corner nodes that cause interference, only one message is sent, say $\mathcal{S}_{(i,j)}(\sqrt{S}, \sqrt{S})$. There is no interference across the clusters due to the edge SBs because edge SBs in shaded clusters cause interference only at MTs belonging to neighboring white clusters, and vice-versa. Within the cluster, linear combinations of messages are sent in a way to zero-force interference by the MBs, and hence the messages are received interference-free. Thus, in each time-slot, $S-1$ messages are sent interference-free, giving a per user DoF of $\frac{(S-1)}{S}$.

□

We note that the achievable schemes in Theorem 10 use only simple zero-forcing and approach the optimal per user DoF of $\frac{1}{2}$ for large S.

Now we consider the hexagonal network with no intra-cell interference and show that when $\sqrt{S} = 3k$, $k \in \mathbb{Z}$, a per user DoF of $\frac{1}{2}$ is achieved. The difference in this case arises because when $\sqrt{S}$ is of the form $3k$, the corner nodes of the shaded (or white) cluster do not cause interference at the corner nodes of the neighboring shaded (or white) clusters as shown in Figure 3.12. We present two achievable schemes that obtain a per user DoF of $\frac{1}{2}$.

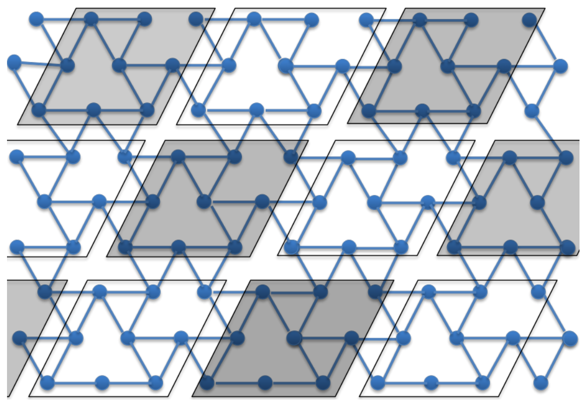


Figure 3.12: The hexagonal sectored cellular network with no intra-cell interference when $\sqrt{S} = 3$. The corner nodes of the same color (shaded or white) do not interfere with each other.

Theorem 11. *For a hexagonal heterogeneous cellular network with no intra-cell interference, where the cluster size is restricted to* $\sqrt{S} = 3k$, $k \in \mathbb{Z}$, *and where* $N \geq min\{\lceil\frac{(\sqrt{S}-2)^2}{2}\rceil + 4(\sqrt{S}-2) + 8, S + 4\}$,

$$\tau_\infty \geq \frac{1}{2}.$$

Proof. We refer to clusters $\mathcal{S}_{(i,j)}$, where $i + j$ is even, as shaded clusters, and the rest as white clusters, as discussed in Section 3.1.1 and shown in Figure 3.6. Interior, edge and corner nodes were introduced in Section 3.1.1 and shown in Figure 3.4.

We propose two achievable schemes: one that uses $N = S + 4$ antennas and one that uses $N = \lceil\frac{(\sqrt{S}-2)^2}{2}\rceil + 4(\sqrt{S}-2) + 8$ antennas.

1. $N = S + 4$
   Backhaul Layer: In odd time-slots, the S SBs corresponding to the shaded clusters receive messages and in even time-slots, the S SBs corresponding to the white clusters receive messages from their respective MBs. We note that the corner SBs in each cluster observe interference, so each MB (i, j) uses additional four antennas to zero-force interference at corner nodes of the clusters $\mathcal{S}_{i-1,j-1}(\sqrt{S}, \sqrt{S})$, $\mathcal{S}_{i+1,j+1}(1, 1)$,

$\mathcal{S}_{i-1,j+1}(\sqrt{S},1)$, $\mathcal{S}_{i+1,j-1}(1,\sqrt{S})$. Over two consecutive time-slots all SBs receive their message interference-free.

Transmission Layer: In even time-slots, the S MTs corresponding to the shaded clusters receive messages and in odd time-slots, the S MTs corresponding to the white clusters receive messages from their respective SBs. In each time-slot, we notice that there is no interference at the MTs. This is because SBs in shaded clusters cause interference only at MTs belonging to neighboring white clusters and vice-versa. Within the cluster, since linear combinations of messages are sent in a way to ZF interference by the MB, the messages are received interference-free. Thus over two consecutive time-slots all MTs receive their message interference-free thus giving a per user DoF of $\frac{1}{2}$.

2. $N = \lceil \frac{(\sqrt{S}-2)^2}{2} \rceil + 4(\sqrt{S}-2) + 8$

   Backhaul Layer:

   - Shaded clusters: In odd time-slots, $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ SBs in the interior of the shaded clusters receive messages and $4(\sqrt{S}-2)$ and eight antennas zero-force interference at the edge nodes and corner nodes in the neighboring white clusters respectively. In even time-slots, $\lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ SBs corresponding to the interior, $4(\sqrt{S}-2)$ edge nodes and the four corner nodes in the exterior of the shaded clusters receive messages. The interference at the four corner nodes of the neighboring shaded clusters is zero-forced by four additional antennas. Note that the exterior SBs do not observe interference because the MBs corresponding to white clusters zero-force interference at these SBs.
   - White clusters: In even time-slots, $\lceil \frac{(\sqrt{S}-2)^2}{2} \rceil$ SBs corresponding to the interior of the white clusters receive messages and $4(\sqrt{S}-2)$ and eight antennas zero-force interference at the edge nodes and corner nodes in the neighboring shaded clusters respectively. In odd time-slots, $\lfloor \frac{(\sqrt{S}-2)^2}{2} \rfloor$ SBs corresponding to the interior, $4(\sqrt{S}-2)$ edge nodes, and the four corner nodes in the exterior of the white clusters receive messages. The interference at the four corner nodes of the neighboring white clusters is zero-forced by four additional antennas. Note that the exterior SBs do not observe interference in the odd time-slots because the MBs

corresponding to neighboring shaded clusters zero-force the interference at these SBs.

Transmission Layer: There is no interference in the transmission layer for interior nodes of a cluster. In even time-slots, the exterior MTs corresponding to the shaded clusters receive messages, and in odd time-slots, the exterior MTs corresponding to the white clusters receive messages from their respective SBs. In each time-slot, we notice that there is no interference between the clusters. This is because SBs in shaded clusters cause interference only at MTs belonging to neighboring white clusters and vice-versa. Within the cluster, linear combinations of messages are sent in such a way as to zero-force the interference by the MB and hence the messages are received interference-free. Over two time-slots all the messages are sent interference-free, thus giving a per user DoF of $\frac{1}{2}$.

□

3.3.1 Time vs. Frequency Duplexing Relays

Since the SBs are half-duplex, they cannot transmit and receive in the same frequency band at the same time. There are two strategies for accommodating this constraint. The first is a frequency-division duplexing (FDD) strategy in which the available frequency band is divided into two equal parts, with the SBs receiving in one half and transmitting in the other. In this case the backhaul and transmission layers can be treated separately, and the puDoF in the total network is half that of the DoF in the transmission layer. The puDoF in locally connected networks is strictly less than one, no matter what the cooperation order M is, and hence the puDoF achievable for the two-layered network is strictly less than $\frac{1}{2}$.

A better strategy for accommodating the half-duplex constraint at the SBs is a time-division duplex (TDD) strategy that we follow above, where the SBs receive and transmit in alternate time slots. In this case also the puDoF in the total network is half of the DoF in the transmission layer, and therefore the maximum achievable puDoF is $\frac{1}{2}$. The key difference from the FDD strategy is that in the TDD strategy we can exploit the fact that not all the SBs are active in a given cluster for the CoMP zero-forcing achievable scheme. The SBs that are inactive for zero-forcing in a given time slot can receive signals in that time slot from the MB serving the cluster, thus utilizing the shared time-frequency resources more efficiently. Using this approach we have shown that one can achieve the maximum possible puDoF of $\frac{1}{2}$ as long as there are a sufficient number of antennas at each of the MBs.

3.3.2 Uplink

For the CoMP schemes discussed in [21] for single-layer networks with a wired backhaul where cooperation is through message sharing on the backhaul, there is no uplink-downlink duality, i.e., the downlink schemes cannot be reversed to provide the same DoF in the uplink. On the other hand if we allow for the sharing of analog signals through the backhaul on the uplink, then the downlink strategies can be reversed to perform interference cancellation on the uplink to achieve the same DoF. In our design of a heterogeneous network, we have a wireless backhaul in the downlink as well as the uplink that enables us to share analog signals through the backhaul. Thus, linear combinations of analog signals can be sent over the backhaul, and uplink-downlink duality holds. In the downlink, we send linear combinations of messages as analog signals to the SBs directly from the MBs and these analog signals are relayed by the SBs to zero-force the interference at the MTs. The uplink can be designed similarly to the downlink, with appropriate combinations of SBs and MTs being scheduled to transmit in different time-slots. Each MB receives a linear combination of analog signals from the active SBs and the messages can be decoded error-free by inverting the channel matrix.[1] Implementing such CoMP reception requires that each MB, the CSI between SBs in its cluster and corresponding MTs is known.

[1]Note that from our assumptions the channel matrix is full rank almost surely.

CHAPTER 4

DYNAMIC SPECTRUM ACCESS

In this chapter, we focus on interference management in a spectrum sharing system in which there is no distinction between users, and in which there is no coordination among the users. The collective performance across all users is more important than that of individual users. This is in contrast to the typical primary/secondary user paradigm in which secondary users bear the responsibility for ensuring priority-based spectrum sharing. We model this system using a multi-user multi-armed bandit (MAB) framework [22]. Our goal is to design an efficient channel access mechanism by managing interference in the system through a decentralized policy across the users.

Multi-arm bandit formulations in stochastic multi-user cognitive radios without user coordination were considered in [23], [24], [25] and [26]. The algorithm in [23] is based on a time-division fair sharing (TDFS) of the best arms between users. Although the algorithm achieves order optimal regret asymptotically, it requires pre-agreement among users and it is assumed that the number of users is fixed and known to all users. The algorithm in [24] does not require any coordination between users and achieves optimal regret asymptotically, but assumes that the number of users is known. The algorithm in [25] combines an ϵ-greedy learning rule with a collision avoidance mechanism, and [26] considers a musical chairs algorithm. Both of these approaches achieve sub-linear regret and do not require knowledge of the number of users. However, it was assumed that the channel parameters are the same for all the users. A stochastic multi-user MAB with user-dependent rewards on channel was considered in [27]. However, the algorithm considers coordination and communication between users via an auction algorithm.

In this work, we focus on two scenarios that have not been previously studied in the multi-user MAB setting for uncoordinated dynamic spectrum access. We assume that the number of users is unknown and that there is no communication between the users. However, we make the mild assumption that the users have access to a shared clock for time synchronization (see also [26, 28, 29]).

We first study a stochastic multi-user MAB where the rewards on the channels are not user-dependent. In our model, all users are treated equally and the reward obtained by each user largely depends on the actions of the other users. When multiple users access the same channel, we allow for a non-zero reward with the assumption that the reward for each user decreases as the number of users on the channel increases. Thus we include the case where there are more users than channels. This is in contrast to the existing approaches, including [25] and [26], which focus on the primary/secondary user paradigm in the scenario where the reward distribution for a user is unknown but fixed. In particular, when multiple users access the same channel they receive zero reward. Hence, all these approaches fail when the number of users is greater than the number of channels.

We assume that the reward on the channel depends on the number of users on the channel and is drawn i.i.d. from a distribution depending on the number of users on the channel. The degradation of the reward as a function of number of users depends on the system, e.g., the distance between the users, and the protocol used for transmission (e.g., hybrid ARQ), and is captured through a reward distribution that depends on the number of users on the channel.

We propose an algorithm and show that if each user employs the algorithm, the systemwide regret is sub-linear in time. The algorithm can be used for any number of users or channels. To the best of our knowledge, we are the first to provide sub-linear regret guarantees without user coordination when the number of users is greater than the number of channels.

In the second scenario, we study the adversarial multi-user MAB framework with user-dependent rewards. The adversarial bandit problem is an important variation of the MAB problem, where no stochastic assumption is made on the generation of rewards. The term "adversarial" refers to the mechanism choosing the sequence of rewards on each arm. If this mechanism is independent of the users actions, then the adversary is said to be *oblivious*. If the mechanism may adapt to the users' past behaviors, then the adversary is said to be *non-oblivious* [22]. The existing literature on adversarial MABs is focused on the single-user case, and a detailed overview of the proposed solutions for the adversarial MAB formulation can be found in [22]. The proposed algorithms in the single-user adversarial setting achieve a sub-linear regret of $O(\sqrt{T})$ over a time horizon T.

We consider multi-user dynamic spectrum allocation without any coordination among the users. We also assume that the rewards on each channel are user-dependent and may vary with time. Such a system is captured through a multi-user adversarial MAB model, particularly when the reward distribution for each channel and user may change over time. We propose an algorithm, and show that if each user employs the algorithm, the systemwide regret is $O(T^{\frac{3}{4}})$ over a time horizon T. To the best of our knowledge, we are the first to consider the multi-user setting for adversarial MABs and to provide sub-linear regret guarantees.

4.1 System Model and Notation

Let K be the number of users in the system. We initially assume that the users have unlimited data for transmission. In a more realistic setting, users may become active or inactive depending on their transmission needs; our dynamic setting covers this scenario. Each user can choose one among M channels for transmission. With M channels and K users attempting to access the spectrum, we assume that each user has prior knowledge of M, but not of K. The assumption of known M is reasonable if the spectrum partition is enforced and fixed. On the other hand, it is not realistic to assume the knowledge of K in an uncoordinated network.

We model the system as a multi-user MAB system with K users and M arms (channels). In each time unit t, let $\mathcal{A}_t^k$ denote the set of channels available to user k. User k chooses a channel $a_t^k \in \mathcal{A}_t^k$ based on the reward history according to a certain policy and receives a reward g_t^k. We assume that $g_t^k \in [0, 1]$, and that each user chooses a channel according to the same algorithm. The reward on each arm depends on the number of users who have chosen the arm. Let $f_t = [f_t(1), \dots, f_t(M)]$ denote the number of users on each channel at time t, where $\sum_{m=1}^{M} f_t(m) = K$. Thus, the reward $g_t^k(a_t^k, f_t(a_t^k))$ received by user k at time t is a function of the channel chosen a_t^k and the number of users on the channel $f_t(a_t^k)$.

4.1.1 Stochastic Setting

We model the system as a stochastic multi-user MAB system with K users and M arms (channels). Each user can choose one among M channels for transmission, where we allow for the possibility that $K \geq M$. We assume that the reward observed is inversely proportional to the number of users transmitting on the same channel. For example, the reward could be the rate achieved by the user on the channel which reduces due to interference from other users accessing the channel. Let $\mu(m, f(m))$ denote the mean reward on channel m when the number of users on the channel is $f(m)$. We assume that each user chooses a channel according to the same policy. We assume that $\mu(m, f(m))$ becomes negligible for some $f(m) = \beta + 1$, where β depends on the system. This restricts the number of users in the system as $\frac{K}{M} \leq \beta$.

In order to ensure that one user does not monopolize a channel for an extended period of time, we impose the following condition. For each user, transmission on a particular channel takes place for a maximum of T_x time units, after which the user releases the channel for at least T_x time units before attempting to access the same channel.

We define the expected regret in the system as

$$\mathbb{E}[R(T)] = T \sum_{i=1}^{M} f^*(i)\mu(i, f^*(i)) - \sum_{t,k} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))]$$

where $f^* = \text{argmax}_f \sum_{i=1}^{M} f(i)\mu(i, f(i))$ corresponds to the optimal number of users on each channel.

To estimate the means on each channel as a function of number of users, we need to impose the following separability condition.

For any $m \in [M]$ and $r, s \in [\beta]$ and some $\epsilon_2 \in (0, 1)$,

$$|\mu(m, r) - \mu(m, s)| \geq 4Mc \exp\left(\frac{K-1}{M-1}\right)\sqrt{\sigma^2 + \epsilon_2}, \tag{4.1}$$

where σ^2 is the variance of the distributions and c is a constant.

4.1.2 Adversarial Setting

In this case, we model the system as an adversarial multi-user MAB with K users and M channels. We further restrict attention to the setting where there are more channels than users in the system i.e., $K \leq M$. We assume that each user chooses a channel according to the same algorithm. For user $k \in [K]$, let $p_t^k = (p_{t+1}^k(1), ..., p_{t+1}^k(M))$ denote the probability vector across the arms, where $p_t^k(m)$ is the probability of choosing arm m at time t. We assume that the adversary chooses different reward for different users for the same channel. Let $g_t^k(a_t^k, f(a_t^k))$ denote the reward observed by user k on choosing channel a_t^k at time t. We assume that if more than one user chooses the same channel, they all receive zero reward. In other words, the users observe zero reward on collision. If there is no collision on the channel, the user observes a reward that is chosen by an adversary. Thus, we set $g_t^k(a_t^k) = 0$ when $f(a_t^k) > 1$.

We adopt the standard notion of pseudo-regret used for adversarial bandits in [22]. The expected total regret in the system until time T is defined as

$$\mathbb{E}[R(T)] = \max_{\mathcal{K}:\mathcal{K}\subseteq[M],|\mathcal{K}|=K} \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i\in\mathcal{K}} g_t^k(i) - \sum_{t=1}^{T}\sum_{k=1}^{K} g_t^k(a_t^k)\right].$$

4.2 Stochastic Setting

In this section, we focus on the stochastic multi-user MAB with user-independent rewards on each channel. We present an algorithm which leads to sub-linear regret, and extend it to the dynamic case.

4.2.1 Algorithm

The algorithm has two phases. The first is an estimation phase during which we estimate the number of users K and $\mu(m, f(m))$, the average mean reward on each channel as a function of the number of users on the channel. The second is an allocation phase where the users arrange themselves in a way that minimizes system regret.

We estimate the number of users by keeping track of the number of colli-

Algorithm 2

for $t = 0$ to T_0 **do**
  $m \sim U(M)$
  if no collision **then**
    $\text{co}_m \leftarrow \text{co}_m + 1$
    $x_1(m) \leftarrow x_1(m) + r(t)$
  else
    append $r(t)$ to $x(m)$
    $\eta_c \leftarrow \eta_c + 1$
  end if
end for
$\hat{K} \leftarrow \min\{1 + \text{round}\left(\frac{\ln(\frac{T_0 - \eta_c}{T_0})}{\ln(1 - \frac{1}{M})}\right), \beta M\}$ and $\hat{\mu}(:, 1) \leftarrow \frac{x_1}{\text{co}}$
if $\hat{K} > M$ **then**
  $\hat{\mu}(m, 2 : \beta) \leftarrow$ Cluster $(x(m))$ for all m
  Calculate $\hat{f}_\epsilon$ from $\hat{\mu}(m, f), \hat{K}$
  Permute$(N_0, T_f + T_x, \infty)$
else
  ch $=$ Alloc$(\hat{M}, T_f + T_x)$ where $\hat{M}$ is set of $\hat{K}$ best channels
  After T_x, choose ch+1 in $\hat{M}$ for next T_x time units
end if

sions similar to [26], with the estimate given by

$$\hat{K} = \min\{1 + \text{round}\left(\frac{\ln(\frac{T_0 - \eta_c}{T_0})}{\ln(1 - \frac{1}{M})}\right), \beta M\}.$$

Algorithm 3 Cluster

Run an α-approximation algorithm for the k-means problem on input X, obtain β means $\nu_1, \ldots, \nu_\beta$
$S_r \leftarrow \{i : |x_i - \nu_r| \leq |x_i - \nu_s|$ for every s$\}$
Return $g(S_r) = \frac{1}{|S_r|} \sum_{i \in S_r} x_i$

We estimate $\mu(m, n)$ separately for each channel based on the reward $x(m)$ observed on the corresponding channel, by clustering the samples using the k-means algorithm. We employ the algorithm Cluster (see Algorithm 3) inspired by [30]. We are interested in finding the centroids of the clusters rather than the correct classification of all the samples. Hence, we use an α-approximation algorithm with a run time T_c to find the estimates the centroids of the cluster and show that we get good estimates with high probability. We consider the approximation algorithm in [31] with a run time $T_c \sim O(T_0)$.

Algorithm 4 Alloc

for $t = 1$ to T **do**
  $a_t \sim U(A)$
  if $\mu(a_t, f(a_t)) \geq \mu(a_t, \hat{f}_\epsilon(a_t))$ **then**
    Choose action $a_\tau = a_t, \quad \forall \tau \geq t$
  end if
end for

Algorithm 5 Permute

$A_1 = [M]$
for $i = 1$ to N_0 **do**
  $q(i) = \text{Alloc}(A_i, T_f + T_x)$;
  $A_i \leftarrow [M] \backslash \{q(i)\}$
end for
while $t \leq T_1$ **do**
  $j = t \mod N_0$
  Choose $q(j)$ for next $\min\{T_1, j(T_x + 1) - 1\}$ rounds
end while

After obtaining estimates for $\hat{\mu}(m, f)$ and $\hat{K}$, the estimate for optimal number of users on each channel $\hat{f}_\epsilon$ can be calculated. We use Alloc (see Algorithm 4) to ensure that each user settles or 'fixes' on a channel m, for which the number of users is less than $\hat{f}_\epsilon(m)$. That is, on finding a channel m with reward greater than $\mu(m, \hat{f}_\epsilon(m))$, the user keeps transmitting on it for at most T_x time units. The system incurs regret until all users have settled on some channels, and we call this duration the *fixing time*. Once all the users have settled on their channels the system does not incur regret. However in our system model, a user can transmit on a channel for at most T_x time units, after which the user must switch. We assume that T_x is fixed for all the users but can vary with time. We use Permute (see Algorithm 5) to construct an efficient allocation for which the regret does not grow with time. We define fixing period as the time during which users 'fix' on a channel and transmit on it for at most T_x sec. In order to avoid systemwide regret every time users have to switch, we fix the ordering of each user after N_0 fixing periods; this can be done for any $N_0 \geq 2$. Our goal is to have each user transmit on all the channels. This is the coupon collector problem with each user having to collect M channels with the expected number of trials $N_0 \sim O(M\text{log}M)$. When $K \leq M$, in order to have efficient allocation so that the regret does not grow with time, after the first epoch, each user switches to the next channel among the set of K best channels.

We fix the fixing period to be of length $T_x + T_f$, where T_f is the expected time taken for all the users to fix on a channel. After N_0 fixing periods, we continue with a time period of length of T_x. We assume that $2 \max_m f^*(m) \leq \sum_m f^*(m)$ to ensure that after every transmitting for T_x time units, each user has other available channels. Note that our algorithm works even when $K \leq M$, in which case it reduces to a version of the algorithm in [26].

4.2.2 Analysis

We investigate the case where $K > M$. We show that if all the users in the system use Algorithm 2, the expected regret is sub-linear in time.

Estimation phase

In the estimation phase, we find estimates for the mean, and the number of users in the system K. More precisely, we find estimates $\hat{\mu}^k(m,n)$ such that $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon$, $\forall k \in [K], \forall m \in [M], n \in [\beta]$, and $\hat{K}$ such that $\hat{K} = K$ with high probability.

Lemma 3. *For any fixed $\epsilon \in (0,1), \delta \in (0,1)$, user k, channel m and number of users on the channel $n \leq \beta$, the estimate $\hat{\mu}^k(m,n)$ obtained after running Algorithm 2 for $T_0 = \left\lceil \frac{32\exp(\frac{K-1}{M-1})M}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \right\rceil$ and the α approximation algorithm for $T_c \sim O(T_0)$ rounds, we have with probability at least $1-\delta$,*

$$|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon.$$

Proof. Let D_1 denote the event that there is at least one combination k, m, n such that $|\hat{\mu}^k(m,n) - \mu(m,n)| \geq \epsilon$ and D_2 denote the event that each player has more than $\frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta}$ observations from distribution with mean $\mu(m,n)$ for each m,n.

$$\begin{aligned}\Pr(D_1) &= \Pr(D_1|D_2)\Pr(D_2)+\Pr(D_1|D_2^c)\Pr(D_2^c)\\ &\le \Pr(D_1|D_2)+\Pr(D_2^c).\end{aligned}$$

It suffices to show that $\Pr(D_1|D_2)\le\frac{\delta}{2}$ and $\Pr(D_2^c)\le\frac{\delta}{2}$. From Lemma 6 in the appendix, we have $\Pr(D_2^c)\le\frac{\delta}{2}$.

$$\Pr(D_1|D_2)\le\sum_{k,m,n}\Pr(|\hat{\mu}^k(m,n)-\mu(m,n)|\ge\epsilon|D_2),$$

where the inequality follows from union bound. To show that $\Pr(D_1|D_2)\le\frac{\delta}{2}$, it suffices to show that $\Pr(|\hat{\mu}^k(m,n)-\mu(m,n)|\ge\epsilon|D_2)\le\frac{\delta}{2MK(\beta+1)}$ which follows from Lemma 8 in the appendix with $\delta\leftarrow\frac{\delta}{2MK(\beta+1)}$ for $n\ge 2$ and follows from Hoeffding's inequality for $n=1$. $\square$

Lemma 4. *For any fixed $\delta\in(0,1)$, user k and channel m, the estimate $\hat{\mu}^k(m,n)$ obtained after running Algorithm 2 for $T_0\ge\lceil\frac{M^2\exp 2(\frac{K-1}{M-1})}{2(0.49)^2}\ln(\frac{2}{\delta})\rceil$, we have with probability at least $1-\delta$,*

$$\hat{K}=K.$$

Proof. Probability of collision for a user at any time is given by

$$p=1-\Pr(\text{No collision})=1-\sum_{\text{channels}}\frac{1}{M}(1-\frac{1}{M})^{K-1}=1-(1-\frac{1}{M})^{K-1}.$$

Let $\hat{p}_t=\frac{\sum_\tau 1\{\text{collision at time }\tau\}}{t}$. We have $E[\hat{p}_t]=p$ and we can use Hoeffding's inequality since collision at each time-slot is independent. Thus if $t\ge\frac{\ln(\frac{2}{\delta})}{2\epsilon_2^2}$, with probability greater than $1-\delta$, we have $|\hat{p_t}-p|\le\epsilon_2$.

We have $\hat{K}=\text{round}(\frac{\ln(1-\hat{p}_t)}{\ln(1-\frac{1}{M})}+1)$ and $K=\frac{\ln(1-p)}{\ln(1-\frac{1}{M})}$. In order to show $\hat{K}=K$, it suffices to show

$$|\hat{K}-K|=|\frac{\ln(\frac{1-\hat{p}_t}{1-p})}{\ln(1-\frac{1}{M})}|\le 0.49,$$

which is equivalent to showing

$$(1-p)(1-(1-\frac{1}{M})^{-0.49})\le\hat{p}_t-p\le(1-p)(1-(1-\frac{1}{M})^{0.49}).$$

It suffices to show

$$\epsilon_2 \leq (1-p)\min\{|(1-(1-\frac{1}{M})^{-0.49})|, |(1-(1-\frac{1}{M})^{0.49})|\}.$$

We have

$$|(1-(1-\frac{1}{M})^{-0.49})| = (1+\frac{1}{M-1})^{0.49} - 1 \geq \frac{0.49}{M-1})$$

and

$$(1-(1-\frac{1}{M})^{0.49}) \geq \frac{0.49}{M},$$

where the inequalities follow from the Bernoulli inequality, $(1+x)^r \leq 1+xr$ for $0 \leq r \leq 1$ and $x \geq -1$.

We have from $(1-\frac{1}{x})^{x-1} \geq \frac{1}{\exp(1)}$ for $x \geq 1$,

$$1-p = (1-\frac{1}{M})^{K-1} \geq \frac{1}{\exp(\frac{K-1}{M-1})}.$$

Hence, we choose $\epsilon_2 \leq \frac{0.49}{M\exp(\frac{K-1}{M-1})}$. □

Allocation phase

In the estimation phase, we obtain estimates $\hat{\mu}^k(m,n)$ such that $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon$, $\forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K}$ such that $\hat{K} = K$ with high probability. We compute $\hat{f}_\epsilon = [\hat{f}_\epsilon(1), \hat{f}_\epsilon(2), \ldots, \hat{f}_\epsilon(M)]$, an estimate of the optimal number of users on each channel from $\hat{K}$ and $\hat{\mu}$.

Here, $\hat{f}_\epsilon = \text{argmax}_f \sum_{i=1}^{M} f(i)\hat{\mu}(i, f(i))$ over all feasible f such that $f \in \mathbb{N} \cup \{0\}^{1\times M}$ and $\sum_{i=1}^{M} f(i) = \hat{K}$.

We now consider a known time τ. Fix $\epsilon = \tau^{-\frac{1}{3}}$ and $\delta = \frac{1}{\tau}$. Using these values, we have $T_0 \sim O(\tau^{\frac{2}{3}}\ln\tau)$ and $T_c = O(T_0)$. Let c_3 be a constant such that $T_c \leq c_3 T_0$. There exists a τ large enough such that we have $\tau \geq T_0 + T_c$. We consider such a τ.

For $\epsilon_i \in (0,1), \forall i \leq r$, let A_{ϵ_i} denote the event that in the ith epoch, $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon_i$, $\forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K} = K$.

We now present the following upper bound which holds during the allocation phase.

Lemma 5. *For any $\tau \geq T_0 + T_c$ such that $\epsilon = \tau^{-\frac{1}{3}}$ and $\delta = \frac{1}{\tau}$ and $\hat{f}_\epsilon$, we have*

$$\sum_{t=T_0+T_c}^{\tau} (\sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon)) - \sum_{k=1}^{K} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon]) \leq N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right).$$

Proof. The allocation phase consists of at most N_0 fixing periods. In each fixing period, the system settles to the configuration $\hat{f}_\epsilon$. After N_0 fixing periods, the system switches between the channels chosen in previous N_0 periods such that the configuration remains $\hat{f}_\epsilon$. The expected time taken for the system to settle to $\hat{f}_\epsilon$ is denoted by T_f. We first calculate the expected fixing time for each user. Let t_f^k denote the fixing time for user k.

Let $\mathcal{M}_t$ denote the set of unfixed arms at time t. Probability of user k being fixed at time t is given by

Pr(User k being fixed)

$$\begin{aligned}
&= \sum_{m \in \mathcal{M}_t} \Pr(\text{user } k \text{ choosing arm } m) \Pr(\text{Being fixed}|\text{arm } m) \\
&= \sum_{m \in \mathcal{M}_t} \frac{1}{M} \Pr(\text{At most } \hat{f}_\epsilon(m) - 1 \text{ users choose arm } m) \\
&= \sum_{m \in \mathcal{M}_t} \frac{1}{M} \sum_{i=0}^{\hat{f}_\epsilon(m)-1} \binom{K-1}{i} \left(\frac{1}{M}\right)^i \left(1 - \frac{1}{M}\right)^{K-1-i} \\
&\underset{(a)}{\geq} \frac{1}{M}\left(1 - \frac{1}{M}\right)^{K-1} = \frac{1}{M}\left(1 - \frac{1}{M}\right)^{(K-1)*(M-1)/(M-1)} \\
&\underset{(b)}{\geq} \frac{1}{M} \frac{1}{\exp(\frac{K-1}{M-1})},
\end{aligned}$$

where (a) follows because we only consider one term in the each of the summations with $i = 0$, and (b) follows from $(1 - \frac{1}{x})^{x-1} \geq \frac{1}{\exp(1)}$ for $x \geq 1$. Thus for any user k, the expected fixing time is given by

$$\mathbb{E}[t_f^k] = \frac{1}{p(\text{User } k \text{ being fixed})} \leq M\text{exp}\left(\frac{K-1}{M-1}\right)$$

and thus the expected time for the system to settle to $\hat{f}_\epsilon$ is given by

$$T_f = \mathbb{E}\left[\max t_f^k\right] \leq \mathbb{E}\left[\sum_{k=1}^{K} t_f^k\right] \leq KM\text{exp}\left(\frac{K-1}{M-1}\right).$$

Let there be N fixing periods till time τ where $N \le N_0$. Each fixing period is of length $T_f + T_x$. Let t_{f_i} denote the fixing time for the system in fixing period $i \in [N]$. Note that $\mathbb{E}[t_{f_i}] = T_f$ for any $i \in [N]$. We now consider $\sum_{t=T_0+T_c+1}^{\tau} (\sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i,\hat{f}_\epsilon)) - \sum_{k=1}^{K} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon])$

$$
\begin{aligned}
&\underset{(a)}{=} \mathbb{E}[\sum_{t=T_0+T_c+1}^{\tau} \sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i,\hat{f}_\epsilon) - \sum_{k=1}^{K} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon]|t_{f_1}, \dots, t_{f_N}] \\
&\underset{(b)}{\le} \mathbb{E}[\sum_{t=T_0+T_c+(i-1)(T_f+T_x)+1}^{T_0+T_c+(i-1)(T_f+T_x)+\min(t_i, T_f+T_x)} \sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i,\hat{f}_\epsilon)|t_{f_1}, \dots, t_{f_N}] \\
&\quad -\mathbb{E}[\sum_{t=T_0+T_c+(i-1)(T_f+T_x)+1}^{T_0+T_c+(i-1)(T_f+T_x)+\min(t_i, T_f+T_x)} \sum_{k=1}^{K} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon])|t_{f_1}, \dots, t_{f_N}] \\
&\underset{(c)}{\le} \mathbb{E}[\sum_{i=1}^{N} \min(t_i, T_f+T_x)(\sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i,\hat{f}_\epsilon))] \le \mathbb{E}[\sum_{i=1}^{N} t_i(\sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i,\hat{f}_\epsilon))] \\
&\underset{(d)}{\le} \mathbb{E}[K\sum_{i=1}^{N} t_{f_i}] = KNT_f \\
&\underset{(e)}{\le} N_0 K^2 M \exp(\frac{K-1}{M-1}).
\end{aligned}
$$

Equality (a) follows from the tower property of expectation. Inequality (b) is because the system is in configuration $\hat{f}_\epsilon$ in each of the N phases after the users settle and after N_0 fixing periods. In each fixing period the system settles within t_{f_i} or reaches the end of fixing period of length $T_f + T_x$. Inequality (c) holds because the reward is non-negative. Inequality (d) holds because the reward is upper bounded by one and $\sum_{i=1}^{M} \hat{f}_\epsilon(i) = K$. Inequality (e) holds because $N \le N_0$ and $T_f \le KM\exp\left(\frac{K-1}{M-1}\right)$.

□

Remark 2. *For the case where $K \le M$, there is no need for clustering. We only need the estimates for $\mu(m, 1)$, and all users individually choose the best K channels. This reduces to the "musical chairs" algorithm and the analysis can be found in [26]. After fixing on a channel during the first fixing period, after every T_x time units, each user switches to the next channel among the K best channels.*

Main result

We now present the upper bound on the conditional expected regret incurred by the users employing Algorithm 2. Let A_ϵ denote the event that $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon, \forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K} = K$.

Theorem 12. *If $\epsilon = \tau^{-\frac{1}{3}}$ and $\delta = \frac{1}{\tau}$, for $\tau \geq T_0 + T_c$, the expected regret for K users using Algorithm 2 with M arms for τ rounds, with parameter $T_0 = \left\lceil \frac{32\exp(\frac{K-1}{M-1})M}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \right\rceil$, $T_c \sim O(T_0)$ and any N_0, is given by*

$$\mathbb{E}[R(\tau)] \leq \tau 2K\epsilon + K(T_0 + T_c) + N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right) + K\delta\tau,$$

i.e., $\mathbb{E}[R(\tau)] \sim O(\tau^{\frac{2}{3}} \ln \tau)$.

Proof. Let A_ϵ denote the event that $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon, \forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K} = K$. Let $\hat{f}_\epsilon$ be the estimate of the optimal number of users on each channel computed from $\hat{K}$ and $\hat{\mu}$.

We first show that

$$\tau \sum_{i=1}^{M} f^*(i)\mu(i, f^*(i)) - \tau \sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon(i)) \leq 2K\epsilon. \tag{4.2}$$

Consider $\sum_{i=1}^{M} f^*(i)\mu(i, f^*(i)) - \sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon(i))$

$$\begin{aligned}
&= \sum_{i=1}^{M} f^*(i)[\mu(i, f^*(i)) - \hat{\mu}(i, f^*(i))] + \\
&\quad \sum_{i=1}^{M} f^*(i)\hat{\mu}(i, f^*(i)) - \sum_{i=1}^{M} \hat{f}_\epsilon(i)\hat{\mu}(i, \hat{f}_\epsilon(i)) + \\
&\quad \sum_{i=1}^{M} \hat{f}_\epsilon(i)[\hat{\mu}(i, \hat{f}_\epsilon(i)) - \mu(i, \hat{f}_\epsilon(i))] \\
&\underset{(a)}{\leq} \epsilon(\sum_{i=1}^{M} f^*(i) + \hat{f}_\epsilon(i)) \\
&\underset{(b)}{\leq} 2K\epsilon.
\end{aligned}$$

Inequality (a) is true since $\hat{f}_\epsilon = \text{argmax} \sum_{i=1}^{M} f(i)\hat{\mu}(i, f(i))$ and hence $\sum_{i=1}^{M} f^*(i)\hat{\mu}(i, f^*(i)) \leq \sum_{i=1}^{M} \hat{f}_\epsilon(i)\hat{\mu}(i, \hat{f}_\epsilon(i))$, and $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon$, $\forall m \in [M], n \in [\beta]$. Inequality (b) is true since $\sum_{i=1}^{M} \hat{f}_\epsilon(i) = \sum_{i=1}^{M} f^*(i) = K$.

The expected regret given $\hat{f}_\epsilon$ is due to regret during the estimation phase as well as the allocation phase. From Lemmas 3 and 4, the estimation phase consists of $T_0 + T_c$ time units where T_c is the time taken for the α-approximation algorithm clustering algorithm to return estimates of the mean. Thus, $K(T_0 + T_c)$ corresponds to the maximum regret accumulated systemwide during the estimation phase. Recall that $T_c \sim O(T_0)$ and $T_0 \sim O(\frac{-\ln\delta}{\epsilon^2}) \sim O(\tau^{\frac{2}{3}} \ln \tau)$. In the allocation phase, the regret in the system is accrued only during the N_0 number of fixing phases. From Lemma 5, the regret in the allocation phase is at most $K^2 M \exp(\frac{K-1}{M-1})$. Let T_f denote the time taken for all the users to fix.

Thus, we have conditional regret

$$
\begin{aligned}
\mathbb{E}[R(\tau)|A_\epsilon] &\leq \tau \sum_{i=1}^{M} f^*(i)\mu(i, f^*(i)) - \sum_{t,k} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon] \\
&= \tau \sum_{i=1}^{M} f^*(i)\mu(i, f^*(i)) - \tau \sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon(i)) + \\
&\quad \tau \sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon) - \sum_{t,k} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon] \\
&\underset{(c)}{\leq} \tau 2K\epsilon + \sum_{t=1}^{T_0+T_c} (\sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon)) - \sum_{k=1}^{K} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon]) + \\
&\quad \sum_{t=T_0+T_c}^{\tau} (\sum_{i=1}^{M} \hat{f}_\epsilon(i)\mu(i, \hat{f}_\epsilon)) - \sum_{k=1}^{K} \mathbb{E}[g_t^k(a_t^k, f_t(a_t^k))|A_\epsilon]) \\
&\underset{(d)}{\leq} \tau 2K\epsilon + K(T_0 + T_c) + N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right).
\end{aligned}
$$

Inequality (c) follows from (4.2). Inequality (d) holds because the reward is non-negative, upper bounded by one and $\sum_{i=1}^{M} \hat{f}_\epsilon(i) = K$, and also from Lemma 5.

We now consider the systemwide unconditional regret. From Lemmas 3 and 4, we have $\Pr(A_\epsilon^c) \leq \delta$.

$$\begin{aligned}\mathbb{E}[R(\tau)] &= \mathbb{E}[R(\tau)|A_\epsilon]\Pr(A_\epsilon) + \mathbb{E}[R(\tau)|A_\epsilon^c]\Pr(A_\epsilon^c) \\ &\leq \tau 2K\epsilon + K(T_0 + T_c) + N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right) + \tau\delta \\ &\sim O(\tau^{\frac{2}{3}} \ln \tau).\end{aligned}$$

□

We now extend the results to the case of any $T \geq \tau^{\frac{2x}{3z}}$ where $x > 1$ and $z < 0.1$, and τ is chosen as before. Each user considers known time τ and runs Algorithm 2. Once the user reaches the end of time τ, the user continues to use Algorithm 2 with a time-period of length $2^x\tau$, where $x > 1$ and so on until time T. The ith epoch of length $i^x\tau$, where $x > 1$. Let T_0^i and T_c^i denote the time taken for estimation and clustering respectively in the ith epoch. Let $\epsilon_i = (i^x\tau)^{-\frac{1}{3}}$ and $\delta_i = \frac{1}{i^x\tau}$ be such that in the ith epoch, with probability greater than $1 - \delta_i$, $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon_i$, $\forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K} = K$.

Algorithm 6

1: **for** $\tau \sum_{i=1}^{r-1} i^x \leq T < \tau \sum_{i=1}^{r} i^x$ **do**
2: Run Algorithm 1 with T_0^i corresponding to $\delta_i \leftarrow \frac{\delta}{i^x}$ and ϵ_i.
3: **end for**

For $\epsilon_i \in (0,1), \forall i \leq r$, let A_{ϵ_i} denote the event that in the ith epoch, $|\hat{\mu}^k(m,n) - \mu(m,n)| \leq \epsilon_i$, $\forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K} = K$.

Theorem 13. *If τ is chosen such that $\tau \geq T_0^1 + T_c^1$, $\epsilon_1 = \tau^{\frac{-1}{3}}$ and $\delta_1 = \frac{1}{\tau}$, the expected systemwide regret after employing Algorithm 6 for T rounds where $T \geq \tau^{\frac{2x}{3z}}$ and $\tau \sum_{i=1}^{r-1} i^x \leq T < \tau \sum_{i=1}^{r} i^x$ with $x > 1$ and $z < 0.1$ is*

$$\mathbb{E}[R(T)] \sim O(T^{z+\frac{2}{3}+\frac{1}{x+1}} \ln T).$$

Proof. T is such that $\tau \sum_{i=1}^{r-1} i^x \leq T < \tau \sum_{i=1}^{r} i^x$. Thus, at time T the users are in the rth epoch.

Using integral to bound the summation, we have $\frac{(r-1)^{x+1}}{x+1} \leq \sum_{i=1}^{r-1} i^x$ which gives us

$$r \leq \left(\frac{(x+1)T}{\tau}\right)^{\frac{1}{x+1}} + 1 \sim O(T^{\frac{1}{x+1}}).$$

Let R_i denote the regret accumulated in epoch i. Let $\epsilon_i = i^{-\frac{x}{3}}$ and $\delta_i = \frac{\delta}{i^x}$, where $x > 1$, $\delta \in (0,1)$. In each epoch i, we employ Algorithm 1 with ϵ_i, δ_i, $T_0^i = \left\lceil \frac{32\exp(\frac{K-1}{M-1})M}{\epsilon_i^2} \ln \frac{2MK\beta(\beta+1)}{\delta_i} \right\rceil$, $T_c^i = O(T_0^i)$. Using Lemmas 3 and 4, in each epoch i we have that with probability greater than $1-\delta_i$, $|\hat{\mu}^k(m,n) - \mu(m,n)| \le \epsilon_i$, $\forall k \in [K], \forall m \in [M], n \in [\beta]$ and $\hat{K} = K$. Thus, for any epoch $i \in [r]$,

$$Pr(A^c_{\epsilon_i}) \le \delta_i. \tag{4.3}$$

From Theorem 12, in each epoch i we have

$$\mathbb{E}[R_i] \le 2i^x \tau K \epsilon_i + K(T_0^i + T_c^i) + N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right) + K\delta_i \tau i^x. \tag{4.4}$$

Since the reward is non-zero and at most one, we have

$$\mathbb{E}[R_i] \le K \times \text{ Length of epoch } i \; = K i^x \tau.$$

Note that $T_c^i \sim O(T_0^i)$ and $T_0^i \sim O(\frac{\ln 1/\delta_i}{\epsilon_i^2})$. Thus, $T_0^i \sim O(i^{\frac{2x}{3}} \ln i)$. The total regret is given as follows:

$$\begin{aligned}
\mathbb{E}[R(T)] &= \sum_{i=1}^{r} \mathbb{E}[R_i] \\
&\le \sum_{i=1}^{r} 2i^x \tau K \epsilon_i + K(T_0^i + T_c^i) + N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right) + K i^x \tau \delta_i \\
&\le \sum_{i=1}^{r} 2i^{\frac{2x}{3}} \tau^{\frac{2x}{3}} K + K(T_0^i + T_c^i) + N_0 K^2 M \exp\left(\frac{K-1}{M-1}\right) + K i^x \tau \frac{1}{\tau i^x} \\
&\sim O(\tau^{\frac{2x}{3}} r^{1+2x/3} + \frac{r \ln r}{\epsilon_r^2} + r) \sim O(\tau^{\frac{2x}{3}} r^{1+\frac{2x}{3}} + \tau^{\frac{2x}{3}} r^{1+\frac{2x}{3}} \ln r + r\tau) \\
&\sim O(\tau^{\frac{2x}{3}} r^{1+\frac{2x}{3}} \ln r) \sim O(\tau^{\frac{2x}{3}} T^{\frac{2}{3}+\frac{1}{x+1}} \ln T)
\end{aligned}$$

If $T \ge \tau^{\frac{2x}{3z}}$, we have $\mathbb{E}[R(T)] \sim O(T^{z+\frac{2}{3}+\frac{1}{(x+1)}} \ln T)$.

□

4.2.3 Dynamic Case

In this subsection, we extend the results to a dynamic system with a changing number of users. Consider a system which starts with K users, and in which users leave the system once they are done with their transmission and new users can take their place. In order to use Algorithm 6 to obtain a sub-linear regret bound, we need to impose some restrictions on the number of users entering and leaving the system until time t, which we denote by Δ_t. We restrict the number of users entering the system Δ_t to be $O(t^\zeta)$ where $\zeta < \frac{5}{6} + 0.5z$ where $z \leq 0.1$.

Let K_t denote the number of active users at time t. In our model, the dynamic scenario also includes the case where K_t can go from greater than M to less than M, and vice-versa. We assume that $K_t \leq \beta M$.

Theorem 14. *The expected systemwide regret after running the Algorithm 6 for T rounds where $T \geq \tau^{\frac{2x}{3z}}$ and $\tau \sum_{i=1}^{r-1} i^x \leq T < \tau \sum_{i=1}^{r} i^x$ is*

$$\mathbb{E}[R(T)] \sim O(T^{\frac{5}{6}+0.5z+\zeta} + T^{\frac{5}{6}+0.5z} \ln T)$$

if $x = \frac{5+3z}{1-3z}$.

Proof. If a user enters or leaves in the middle of an epoch, regret is accumulated till the end of the epoch since the estimates for f, K correspond to the system with number of users at the beginning of the epoch. Thus, the regret due to the user entering or leaving the system is upper bounded by the length of the epoch.

In epochs where no users enter the system, the regret can be bound by Theorem 13, and in epochs with new users, the regret accumulates through the entire epoch.

Until epoch r, let $E_r = \{$ Epoch i: Epoch i has at least one user entering or leaving the system $\}$. Note that $|E_r| \leq \Delta_T$.

Expected regret in epochs with change,

$$\begin{aligned}
\sum_{i \in E_r} \mathbb{E}[R_i] &\leq \beta M \sum_{i \in E_r} \text{Length of epoch } i \\
&\leq \beta M |E_r| r^x \tau \\
&\leq \beta M \Delta_T r^x \tau \sim O(T^{\frac{x}{x+1}+\zeta}).
\end{aligned}$$

The regret up to time T bounded as follows:

$$\begin{aligned}
\mathbb{E}[R(T)] &\leq \sum_{i\in[r]\setminus E_r}\mathbb{E}[R_i] + \sum_{i\in E_r}\mathbb{E}[R_i] \\
&\leq \sum_{i\in[r]\setminus E_r}\mathbb{E}[R_i] + \beta M\Delta_T r^x \\
&\sim O(T^{\frac{2}{3}+z+\frac{1}{(x+1)}}\ln T + T^{\frac{x}{x+1}+\zeta}).
\end{aligned}$$

If set $x = \frac{5+3z}{1-3z}$, we have $O(T^{\frac{5}{6}+0.5z+\zeta} + T^{\frac{5}{6}+0.5z}\ln T)$. Thus, if Δ_T is $O(T^{\zeta})$, with $\zeta < \frac{1}{6} + 0.5z$, we have sub-linear regret. $\square$

4.2.4 Experiments

In this section, our goal is to validate the performance of the estimation phase in the algorithm and observe how the performance in the allocation phase suffers due to use of the estimated values.

We consider a system with $K = 10$ users and $M = 6$ channels for a fixed time horizon $\tau = 11000$. We set $T_0 = 1000$, $T_x = 1000$ time units and $N_0 = 5$ and repeat the experiment 100 times and consider the average accumulated regret. The value of β is set to 3, and the reward distributions are chosen to be uniform with a variance of 0.01, and means between 0 and 1 given below:

$$\mu = \begin{bmatrix} 1 & 0.49 & 0.1 & 0.005 \\ 0.98 & 0.42 & 0.13 & 0.002 \\ 0.97 & 0.5 & 0.12 & 0.009 \\ 1 & 0.48 & 0.009 & 0.008 \\ 0.92 & 0.43 & 0.1 & 0.001 \\ 0.9 & 0.44 & 0.1 & 0.001 \end{bmatrix}.$$

We compare the performance of Algorithm 2 with the estimated values of μ and K with Algorithm 2 with the true parameter values. We also show how the estimates change with number of iterations in the estimation phase T_0. We used the in-built MATLAB `kmeans` function for clustering.

From Figure 4.1, we see that the accumulated regret grows with time during the estimation phase and remains constant after N_0 fixing periods. Also, there is no noticeable difference between Algorithm 2 with the true parameter values and the one with the estimated values. This follows because

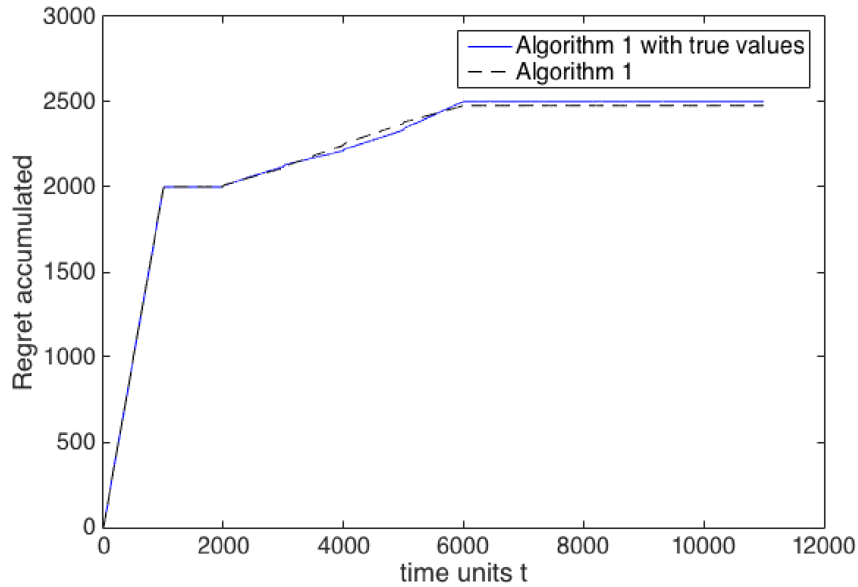


Figure 4.1: Accumulated regret as a function of time.

the estimates of K and the mean converge to the true values within a few iterations as shown in Figures 4.2 and 4.3, and we have the correct estimate for the optimal number of users on each channel.

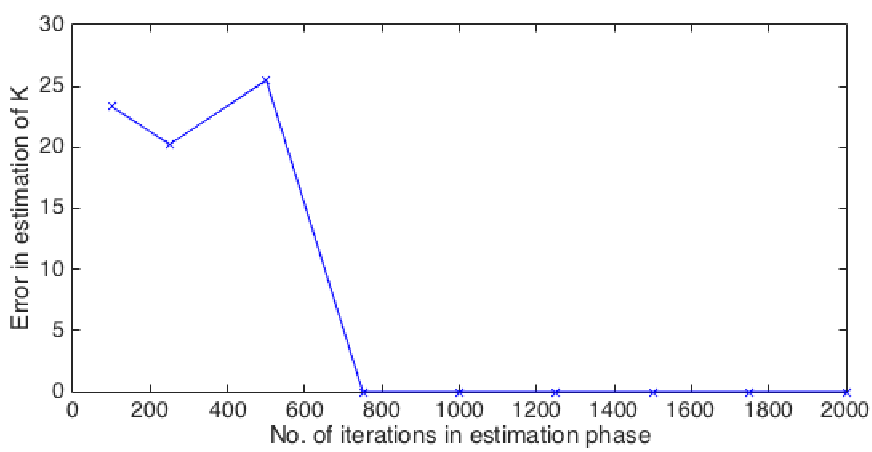


Figure 4.2: Error in the estimation of number of users K.

4.3 Adversarial Setting

In this section, we consider the adversarial multi-user MAB model with user-dependent rewards on each channel. We present an algorithm that leads to sub-linear regret, and extend it to the dynamic case.

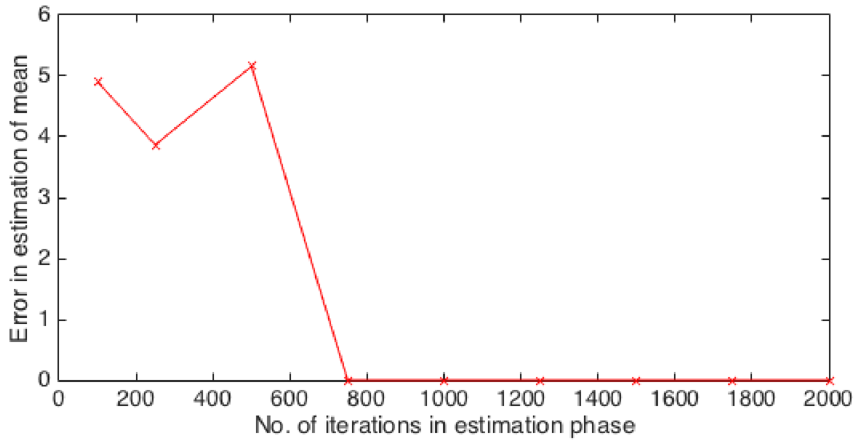


Figure 4.3: Error in the estimation of the mean.

4.3.1 Single-user MAB

We consider the Exp3.P algorithm described in [22] for a single-user MAB in an adversarial setting. We modify the algorithm so that the user chooses an arm and updates the probability vector only in a few time units. This modification is useful in the multi-user case, where the users may not choose an arm in each time unit due to possible collisions. We now present a modified version of the Exp3.P algorithm, in which a new arm is chosen and the probability is updated at time units $t_1, t_2, \ldots, t_n$ such that $n \leq T$ and $\alpha = \max_{j \in [n-1]} t_{j+1} - t_j$. For each $j \in [n]$, we consider the reward over the time-period $t_{j+1} - t_j$, with the reward being normalized to lie between 0 and 1.

Theorem 15. *The expected regret of Modified Exp3.P algorithm (Algorithm 7) until time T is given by*
$\mathbb{E}\left[\sum_{t=1}^{T} g_t(m) - g_t(a_t)\right]$

$$\leq \max_{m \in [M]} \mathbb{E}\left[\sum_{t=1}^{T} (g_t(m) - g_t(a_t))\right] \leq \alpha\sqrt{n} h(M), \tag{4.5}$$

where $h(M) = 5.15\sqrt{M \ln M} + \sqrt{\frac{M}{\ln M}}$, and does not depend on T and $n \leq T$.

Proof. We have

$$\mathbb{E}\left[\sum_{t=1}^{T} (g_t(m) - g_t(a_t))\right] \leq \alpha \mathbb{E}\left[\sum_{j=1}^{n} (g'_j(m) - g'_j(a_j))\right], \tag{4.6}$$

where $g'_j(m) = \frac{\sum_{t_j \leq t \leq t_{j+1}} g_t(m)}{t_{j+1} - t_j}$. Using (4.6), and noting that until time T we

Algorithm 7 Modified Exp3.P

1: $\phi = \sqrt{\frac{\ln M}{Mn}}, \quad \eta = 0.95\sqrt{\frac{\ln M}{Mn}}$ and $\gamma = 1.05\sqrt{\frac{M \ln M}{n}}$.
2: Initial probability distribution $p_0 = (\frac{1}{M}, \ldots, \frac{1}{M})$.
3: **for** $j = 1, \ldots, n$ **do**
4: $a_j \sim p_j$, remain on arm for next $t_{j+1} - t_j$ time units
5: Compute reward as $g'_j(i) = \frac{\sum_{t_j \le t \le t_{j+1}} g_t(i)}{t_{j+1} - t_j}$ and the estimated gain for each arm as

$$\tilde{g}_j(i) = \frac{g'_j(i)\mathbb{1}_{a_j = i} + \phi}{p_j(i)}$$

and update the cumulative gain $\tilde{G}_j(i) = \sum_{s=1}^{j} \tilde{g}_s(i)$
6: Calculate $p_{j+1} = (p_{j+1}(1), ..., p_{j+1}(M))$ where

$$p_{j+1}(i) = (1-\gamma)\frac{\exp(\eta \tilde{G}_j(i))}{\sum_{m=1}^{M} \exp(\eta \tilde{G}_j(m))} + \frac{\gamma}{M}$$

7: **end for**

consider n time units, the proof follows from the regret bound for Exp3.P given in [22]. $\square$

4.3.2 Multi-user MAB: Algorithm

We now consider the multi-user adversarial bandits under a known finite horizon T, and propose an algorithm which when employed by all users independently leads to sub-linear regret.

In a multi-user adversarial system, every time t that a user k chooses an arm according to a certain probability distribution p_t^k to randomize against the adversary, there is a possibility for collision with other users. Hence there is a need for a collision resolution mechanism, so that the regret does not grow linearly with time. Instead of choosing an arm every time unit, a user chooses an arm only a sub-linear number of times until T (e.g., T^y where $y < 1$). The goal is to randomize enough times to counteract the adversary, while making sure that the regret due to collisions does not become large.

We propose an algorithm (Algorithm 8) that combines the modified Exp3.P algorithm (Algorithm 7) with a collision resolution mechanism with $y < 1$. In the analysis in Section 4.3.3, we pick $y = \frac{1}{2}$ which is large enough to maintain the sub-linear regret achieved by the modified Exp3.P algorithm but small enough so that the regret due to collisions is sub-linear as well.

In every time-interval of length T^{1-y}, we first have a collision resolution phase. Each user chooses a channel with probability p_t^k. A user settles or fixes on a channel if at any time the user finds a channel without collision. Once a user settles on a channel, the user keeps transmitting on the channel until the end of the time-interval of length T^{1-y}. The system incurs regret until all K users have settled on K channels, and we call this duration the *fixing time*. The remaining part of the algorithm corresponds to each of the K users employing the modified Exp3.P algorithm, where they choose a channel once every T^y time units.

Algorithm 8

$\phi = \sqrt{\frac{\ln M}{MT^y}}$, $\quad \eta = 0.95\sqrt{\frac{\ln M}{MT^y}}$ and $\gamma = 1.05\sqrt{\frac{M \ln M}{T^y}}$.

The initial probability distribution $p_0^k = (\frac{1}{M}, \ldots, \frac{1}{M})$

for $t =$ multiples of $\frac{T}{T^y}$ **do**

  for $t' = 1$ to T^{1-y} **do**

    $a_{t'}^k \sim p_t^k$

    if no collision **then**

      break

    end if

  end for

  Choose action $a_{t'}^k$ for next $T^{1-y} - t'$ time units

  Compute reward as $g_t'^k(i) = \frac{\sum g_t^k(i)}{T^{1-y}-t'}$ and the estimated gain for each arm as

$$\tilde{g}_t^k(i) = \frac{g_t'^k(i)\mathbb{1}_{a_{t'}^k=i} + \phi}{p_t^k(i)}$$

  and update the cumulative gain $\tilde{G}_t'^k(i) = \sum_{s=1}^{t} \tilde{g}_s^k(i)$

  Calculate $p_{t+1}^k = (p_{t+1}^k(1), ..., p_{t+1}^k(M))$ where

$$p_{t+1}^k(i) = (1-\gamma)\frac{\exp(\eta \tilde{G}_t^k(i))}{\sum_{m=1}^{M} \exp(\eta \tilde{G}_t^k(m))} + \frac{\gamma}{M} \tag{4.7}$$

end for

4.3.3 Multi-user MAB: Analysis

In this subsection, we first consider the regret due to the collision resolution phase, then the regret due to the modified Exp3.P part of Algorithm 8, and then combine them to find an upper bound on the systemwide regret incurred when each user independently employs Algorithm 8.

Regret during collision resolution

Theorem 16. *The expected regret accumulated by the system during a collision resolution phase is upper bounded by*

$$\frac{K^2 M^K}{\gamma} \leq \frac{K^2 M^K T^{\frac{y}{2}}}{\sqrt{M \ln M}}.$$

Proof. We first note from equation (4.7) that the probability of choosing any channel by any user is at least $\frac{\gamma}{M}$. Let $\rho_t^k = \max_m p_t^k(m)$, which implies that $\rho_t^k \geq \frac{1}{M}$. Let "maximal" refer to the channel that has the highest probability of being chosen by that particular user. Thus, each user can be associated with one channel such that probability of choosing it is greater than $\frac{1}{M}$. Since $K \leq M$, for each user, there exists at least one channel such that it is not the maximal channel for any of the remaining $K-1$ users. Note that even when some users fix or settle on a channel, and there are both unfixed channels and unfixed users in the system, we can still find an unfixed channel such that it is not the maximal channel for the remaining unfixed users.

Based on the above discussion, we define the event B_k to be the event where all unfixed users except user k choose their maximal arm, and user k chooses an unfixed arm that is not the maximal arm for any other unfixed users.

Let $\mathcal{M}_t$ denote the set of unfixed arms at time t. The probability of any user k being fixed at time t is given by

Pr{User k being fixed}

$$\begin{aligned}
&= \sum_{m \in \mathcal{M}_t} \Pr\{\text{User } k \text{ is the only unfixed user on arm } m\} \\
&\geq \Pr(B_k) \\
&\geq (\Pi_{i \in [K], i \neq k} \rho_t^i) \min_{m \in \mathcal{M}t} p_t^k(m) \\
&\geq \frac{\gamma}{M} (\frac{1}{M})^{K-1} = \frac{\gamma}{M^K}.
\end{aligned}$$

The remainder of the proof follows similarly to the proof of Theorem 5. □

Regret due to modified Exp3.P

We now bound the regret incurred by the users using Algorithm 8 during the time the users are not in the collision resolution phase. This corresponds to each of the K users independently employing the modified Exp3.P algorithm introduced in subsection 4.3.1.

In Algorithm 8, when the users are not in the collision resolution phase, each user employs modified Exp3.P with $n = T^y$ and $\alpha = T^{1-y}$. Using the result of Theorem 15 for K users, for any distinct set $\mathcal{K} \subseteq [M]$ consisting of K arms,

$$\mathbb{E}\left[\sum_{t \notin \text{coll. phase}} \left(\sum_{i \in \mathcal{K}} g_t^k(i) - \sum_{k=1}^{K} g_t^k(a_t^k)\right)\right] \leq KT^{1-\frac{y}{2}} h(M).$$

Thus,

$$\max_{\mathcal{K}} \mathbb{E}\left[\sum_{t \notin \text{coll. phase}} \left(\sum_{i \in \mathcal{K}} g_t^k(i) - \sum_{k=1}^{K} g_t^k(a_t^k)\right)\right] \leq h(M)KT^{1-\frac{y}{2}}, \tag{4.8}$$

where $h(M) = 5.15\sqrt{M \ln M} + \sqrt{\frac{M}{\ln M}}$, and does not depend on T.

Main result

We now present the upper bound on the expected regret incurred by the users employing Algorithm 8.

Theorem 17. *The expected regret of K users using Algorithm 8 with M arms for T time units is given by*

$$\mathbb{E}[R(T)] \leq T^{\frac{3}{4}} h'(M, K),$$

where $h'(M, K) = K\left(5.15\sqrt{M \ln M} + \sqrt{\frac{M}{\ln M}} + \frac{KM^K}{\sqrt{M \ln M}}\right)$, and does not depend on T. Thus, $\mathbb{E}[R(T)] \sim O(T^{\frac{3}{4}})$.

Proof. The expected regret is due to collision resolution phase as well as the modified Exp3.P algorithm which is played a sub-linear number of times. Let

T_f denote the time taken for all the users to fix.

$$\begin{aligned}
\mathbb{E}[R(T)] &\leq T^y K \mathbb{E}[T_f] + (T^{1-y} - \mathbb{E}[T_f]) h(M) T^{\frac{y}{2}} \\
&\leq \frac{K^2 M^K}{\sqrt{M \ln M}} T^{\frac{3y}{2}} + K T^{1-\frac{y}{2}} h(M) \\
&\sim O(T^{\frac{3y}{2}} + T^{1-\frac{y}{2}}),
\end{aligned}$$

where the inequalities follow from Theorem 16 and equation (4.8), and $h(M) = 5.15\sqrt{M \ln M} + \sqrt{\frac{M}{\ln M}}$. If we choose y such that $\frac{3y}{2} = 1 - \frac{y}{2}$, we have $y = \frac{1}{2}$ which gives us

$$\mathbb{E}[R(T)] \leq T^{\frac{3}{4}} K \left(\frac{K M^K}{\sqrt{M \ln M}} + h(M) \right).$$

□

4.3.4 Unknown Time Horizon

In this subsection, we extend the results to the case of unknown time horizon. Each user considers some known time τ greater than the expected fixing time for the system and runs Algorithm 8. Once the user reaches the end of time τ, the user continues to use Algorithm 8 with a time-period of length 2τ. In this way when the user reaches the end of the previous time-period, the user doubles it and continues with Algorithm 8. Let T be such that $\tau + 2\tau + \ldots + 2^r\tau \leq T \leq \tau + 2\tau + \ldots + 2^{(r+1)}\tau$, equivalently $2^{(r+1)}\tau \leq T + \tau < 2^{(r+2)}\tau$.

Algorithm 9

1: **for** $(2^{(r+1)} - 1)\tau \leq T < (2^{(r+2)} - 1)\tau$ **do**
2: Run Algorithm 1 with time-period $2^{r+1}\tau$
3: **end for**

Theorem 18. *The expected regret from using Algorithm 9 for T time units where $(2^{(r+1)} - 1)\tau \leq T < (2^{(r+2)} - 1)\tau$ is*

$$\mathbb{E}[R(T)] \leq h'(M,K) \frac{(2(T+\tau))^{\frac{3}{4}}}{2^{\frac{3}{4}} - 1},$$

where $h'(M,K) = K\left(5.15\sqrt{M\ln M} + \sqrt{\frac{M}{\ln M}} + \frac{KM^K}{\sqrt{M\ln M}}\right)$ *and does not depend on* T. *Thus,* $\mathbb{E}[R(T)] \sim O(T^{\frac{3}{4}})$.

Proof. We have $2^{(r+1)}\tau \leq T+\tau$. Using Theorem 17, the regret up to time T bounded as follows:

$$\begin{aligned}
\mathbb{E}[R(T)] &\leq h'(M,K)(\tau^{\frac{3}{4}} + (2\tau)^{\frac{3}{4}} + \ldots + (2^{r+1}\tau)^{\frac{3}{4}}) \\
&= h'(M,K)\tau^{\frac{3}{4}}\frac{(2^{(r+2)\frac{3}{4}} - 1)}{2^{\frac{3}{4}} - 1} \\
&\leq h'(M,K)\frac{(2(T+\tau))^{\frac{3}{4}} - \tau^{\frac{3}{4}}}{2^{\frac{3}{4}} - 1}.
\end{aligned}$$

□

Note that each user only needs knowledge of K in order to fix on an initial τ such that $\tau \geq \mathbb{E}T_f$, where T_f is the fixing time for all the users in the system. Furthermore, τ can be chosen even without the knowledge of K by simply replacing K by M, and the analysis follows because $K \leq M$.

4.3.5 Dynamic Case

In this subsection, we extend the results to a dynamic system with a changing number of users. Consider a system which starts with K users, and in which users leave the system once they are done with their transmission. It is easy to see that Algorithm 9 in this case leads to systemwide regret of the order $O(T^{\frac{3}{4}})$ over a time horizon T.

Let us now consider a dynamic system where users enter and leave the system over time. In order to use Algorithm 9 to obtain a sub-linear regret bound, we need to impose some restrictions on the number of users that have entered the system until time t, which we denote by κ_t. It is easy to see that the number of epochs in which users enter the system must be sub-linear in time to have sub-linear regret in the system. We restrict the number of users entering the system κ_t to be $O(t^{\zeta})$ where $\zeta < \frac{1}{2}$.

Let K_t denote the number of active users at time t. Note that even in the dynamic scenario, we still retain the assumption of having $K_t \leq M$ in the system.

Theorem 19. *The expected systemwide regret from using Algorithm 9 for* T *time units where* $(2^{(r+1)}-1)\tau \leq T < (2^{(r+2)}-1)\tau$ *with the number of users* *entering the system $\kappa_T \sim O(T^\zeta)$, with $\zeta < \frac{1}{2}$, is given by*

$$\mathbb{E}[R(T)] \leq h'(M,M)\frac{(2(\tau+T))^{\frac{3}{4}}}{2^{\frac{3}{4}}-1} + M\kappa_T T^{\frac{1}{2}},$$

where $h'(M,M) = M\left(5.15\sqrt{M\ln M} + \sqrt{\frac{M}{\ln M}} + \frac{M^{M+1}}{\sqrt{M\ln M}}\right)$ and does not depend on T. Thus, $\mathbb{E}[R(T)] \sim O(T^{\frac{3}{4}} + \kappa_T T^{\frac{1}{2}})$.

Proof. We have $2^{(r+1)}\tau \leq \tau + T$. In epochs where no users enter the system, the regret can be bound by Theorem 18, and in epochs with new users, the regret accumulates through the entire epoch. The epoch length is upper bounded by $(2^{(r+1)}\tau)^{\frac{1}{2}}$, since $y = \frac{1}{2}$ from Theorem 12.

Until epoch r, let $E_r = \{$ Epoch i : Epoch i has at least one user entering the system$\}$. Note that $|E_r| \leq \kappa_T$.

Let R_i denote the regret accumulated in epoch i. Expected regret in epochs with change,

$$\begin{aligned}
\sum_{i\in E_r}\mathbb{E}[R_i] &\leq M\sum_{i\in E_r}\text{Length of epoch } i \\
&\leq M|E_r|(2^{(r+1)}\tau)^{\frac{1}{2}} \\
&\leq M\kappa_T(2^{(r+1)}\tau)^{\frac{1}{2}}.
\end{aligned}$$

The regret up to time T bounded as follows:

$$\begin{aligned}
\mathbb{E}[R(T)] &\leq \sum_{i\in[r]\setminus E_r}\mathbb{E}[R_i] + \sum_{i\in E_r}\mathbb{E}[R_i] \\
&\leq h'(M,M)\frac{(2(\tau+T))^{\frac{3}{4}}}{2^{\frac{3}{4}}-1} + M\kappa_T(2^{r+1}\tau)^{\frac{1}{2}} \\
&\leq h'(M,M)\frac{(2(\tau+T))^{\frac{3}{4}}}{2^{\frac{3}{4}}-1} + M\kappa_T(\tau+T)^{\frac{1}{2}}.
\end{aligned}$$

Thus, if κ_T is $O(T^\zeta)$, with $\zeta < \frac{1}{2}$, we have sub-linear regret. □

4.3.6 Experiments

In this section, we illustrate the performance of our algorithm in a simple adversarial setting. We consider a non-oblivious adversary, i.e., an adversary whose rewards do not depend on the users' reward history.

We consider a system with known time-horizon T, fixed number of users $K = 4$ users and $M = 7$ channels. We set $T = 160000$, which gives us $T^{\frac{1}{2}} = 400$ time units, $\phi = 0.026$, $\eta = 0.025$ and $\gamma = 0.194$ in Algorithm 8. The reward distributions for the channels are drawn i.i.d. from the uniform distribution $[a, 1]$ where a for each channel at each time unit is drawn i.i.d. from the uniform distribution $[0.2, 1]$.

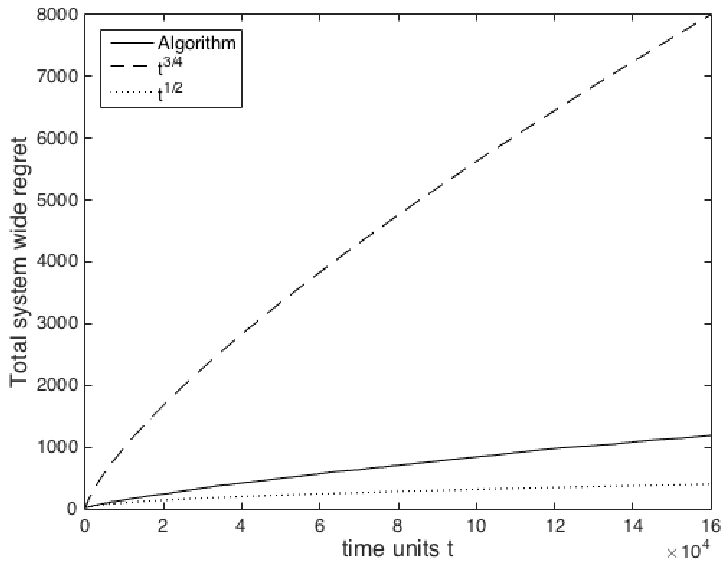


Figure 4.4: Accumulated regret as a function of time.

We repeat the experiment 100 times and consider the average accumulated regret with time. From Figure 4.4, we see that the regret grows with time at a rate much lower than $T^{\frac{3}{4}}$, but higher than $T^{\frac{1}{2}}$, the expected regret in the single-user case.

Remark 3. *We note that Algorithm 8 can be used for a stochastic multi-user MAB with user-dependent rewards to achieve a sub-linear regret of order $O(T^{\frac{3}{4}})$. While the regret is much higher than in [27], our algorithm does not rely on communication between the users and can also deal with a dynamic number of users in the system.*

Remark 4. *In the adversarial case, there is randomization in the selection of a channel, with $\tau^{\frac{1}{2}}$ being equivalent to T_x, and hence each user does not transmit on a channel for a very long time. Thus, fairness is achieved without enforcing a strict duration T_x for each transmission.*

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

We studied interference management in various wireless networks. We first focused on interference management through cooperative transmission in the downlink of a cellular network in Chapters 2 and 3. In Chapter 2, we studied the potential gains offered by cooperative transmission in the downlink of a practically relevant hexagonal sectored cellular network, under an average backhaul load constraint. We showed that DoF gains can be achieved using cooperative transmission under the average backhaul load constraint B by proposing achievable schemes for general integer values of B. The proposed schemes are simple zero-forcing schemes with a flexible message assignment that achieve the information-theoretic upper bound of the per user DoF when cooperation is not allowed. Further, in order to achieve this bound, there is neither need to increase the backhaul load beyond an average of one message per transmitter, nor to use interference alignment. We also showed that $\tau_c^{\text{zf}}(M=1) < \tau_c(M=1)$, i.e., interference avoidance schemes cannot achieve the information theoretic upper bound of $\frac{1}{2}$, a DoF value that *can* be achieved with zero-forcing *cooperative* transmission and no extra backhaul load. Further, we provided a useful upper bound on the per user DoF achievable through cooperative zero-forcing with small values of the backhaul load $B < 5$, $\tau_c^{\text{avg,zf}}(B) \leq \frac{5+B}{10}$. In order to obtain a tight bound on the per user DoF for zero-forcing schemes for any backhaul load constraint B, we formulated the general problem of finding the maximum per user DoF as an optimization problem.

In Chapter 3, we considered a heterogeneous cellular network consisting of MBs, SBs and MTs with a wireless backhaul layer, and with the SBs acting as half-duplex relays. We analyzed the per user DoF first for a linear heterogeneous cellular network, and then extended the results to a more general and practical heterogeneous hexagonal cellular network. We proposed simple zero-forcing schemes that use joint processing to cancel the interference at the MTs. A novel feature of our approach is that appropriate linear combinations of the messages are sent to the SBs, rather than sending multiple messages, thus avoiding overloading the backhaul. In the linear network, our schemes achieve the optimal puDoF of $\frac{1}{2}$, while in the hexagonal network, our schemes approach the optimal puDoF from below. The insights from this work can also be used to design the uplink in a similar fashion, since uplink-downlink duality holds for the proposed achievable schemes.

It is important to note that the conclusions in this work, rely on the assumption that accurate channel state information (CSIT) is available at the transmitters. Recently, the problem of interference management through cooperative transmission has been studied with weak and no CSIT in [32–37]. In [34], it was shown that significant gains could be achieved through a flexible cell association strategy that does not constrain availability of the i^{th} message to only the i^{th} transmitter. In [38], it was shown that cooperative transmission cannot lead to a per user DoF gain in large Wyner's linear networks with no CSIT, when restricted to linear cooperation schemes. It is of interest to extend the work in [32–37] to study interference management using cooperative transmission with weak and no CSIT.

In Chapter 4, we study a decentralized spectrum sharing system, where we proposed efficient algorithms to reduce interference between the users. We modeled the dynamic spectrum allocation problem as a multi-user MAB with no communication among the users. We first considered a stochastic MAB model with rewards on the channel being the same for all users, and then an adversarial MAB model with user-dependent rewards. We showed that the proposed algorithms in both scenarios achieve sub-linear regret. We also extended our algorithms to the dynamic case and showed that the algorithms continue to achieve sub-linear regret. It is of interest to develop algorithms in other variants of the multi-user MAB setting. For example, one might investigate a system with user-dependent rewards, under the stochastic as well as the adversarial settings, without any user communication, when there are more users than channels in the system.

APPENDIX A

CLUSTERING BOUNDS

We present a lemma that ensures a certain number of observations from each distribution during the estimation phase of length T_0.

Lemma 6. *If* $T_0 = \left\lceil \frac{32\exp(\frac{K-1}{M-1})M}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \right\rceil$, *then all users using Algorithm 2 have at least* $\frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta}$ *observations of each reward distribution on each arm with probability greater than* $1 - \frac{\delta}{2}$.

Proof. Let $A_{k,m,n}(t) = I\{\text{player } k \text{ observed arm } m \text{ with } n \text{ users at round } t\}$. Note that for any round t and any k, m, n we have that

$$\Pr(A_{k,m,n}(t) = 1) = \frac{1}{M}\binom{K-1}{n-1}\left(1 - \frac{1}{M}\right)^{K-n}\left(\frac{1}{M}\right)^{n-1}$$

$\implies \mathbb{E}[A_{k,m,n}(t)] = \frac{1}{M}\binom{K-1}{n-1}\left(1-\frac{1}{M}\right)^{K-n}\left(\frac{1}{M}\right)^{n-1} \geq \frac{1}{M}\left(1-\frac{1}{M}\right)^{K-1} \geq \frac{1}{M\exp(\frac{K-1}{M-1})}$ for all $M > 1$, where the last inequality follows from $(1-\frac{1}{x})^{x-1} \geq \frac{1}{\exp(1)}$ for $x \geq 1$.

We have
$\Pr\left(\exists k, m, n \text{ s.t. } \sum_{t=1}^{T_0} A_{k,m,n}(t) \leq \frac{1}{2}T_0\mathbb{E}[A_{k,m,n}(t)]\right)$

$$\begin{aligned}
&\leq \sum_k \sum_m \sum_n \Pr\left(\sum_{t=1}^{T_0} A_{k,m,n}(t) \leq \tfrac{1}{2}T_0\mathbb{E}[A_{k,m,n}(t)]\right) \\
&\leq \sum_k \sum_m \sum_n \exp\left(\frac{-\frac{1}{4}T_0\mathbb{E}[A_{k,m,n}(t)]}{2}\right) \\
&= K(\beta+1)M\exp\left(\frac{-\frac{1}{4}T_0\mathbb{E}[A_{k,m,n}(t)]}{2}\right),
\end{aligned}$$

where the first inequality follows from union bound and the second inequality follows from Chernoff bound. Note that for a particular k, m and n, $A_{k,m,n}$ is i.i.d. across t, since all users are choosing channels uniformly at random.

In order for this probability to be upper bounded by $\frac{\delta}{2}$ we need:

$$K(\beta+1)M\exp\left(\frac{-\frac{1}{4}T_0\mathbb{E}\left[A_{k,m,n}(t)\right]}{2}\right) < \frac{\delta}{2}$$
$$\implies T_0 > \frac{1}{8\mathbb{E}\left[A_{k,m,n}(t)\right]}\ln\left(\frac{2K(\beta+1)M}{\delta}\right).$$

We have shown that if $T_0 > \frac{1}{8\mathbb{E}\left[A_{k,m,n}(t)\right]}\ln\left(\frac{2K(\beta+1)M}{\delta}\right)$ then with probability greater than $1-\frac{\delta}{2}$ we have that $\forall k,m,n$

$$\sum_{t=1}^{T_0} A_{k,m,n}\left(t\right) > \frac{1}{2}T_0\mathbb{E}\left[A_{k,m,n}(t)\right].$$

We also need the total number of observations each player has of each arm to be at least $\frac{16}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}$, i.e.

$$\sum_{t=1}^{T_0} A_{k,m,n}\left(t\right) > \frac{1}{2}T_0\mathbb{E}\left[A_{k,m,n}(t)\right] \geq \frac{16}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}$$
$$\implies T_0 \geq \frac{2}{\mathbb{E}\left[A_{k,m,n}(t)\right]}\frac{16}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}.$$

So we have two constraints on T_0, which gives us:

$$T_0 = \left\lceil \max\left\{\frac{1}{8\mathbb{E}\left[A_{k,m}(t)\right]}\ln\left(\frac{2K(\beta+1)M}{\delta}\right), \frac{32}{\epsilon^2\mathbb{E}\left[A_{k,m}(t)\right]}\ln\frac{2MK\beta(\beta+1)}{\delta}\right\}\right\rceil,$$

which can be further simplified to

$$T_0 = \left\lceil \frac{32\exp(\frac{K-1}{M-1})M}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}\right\rceil.$$

□

A.1 Clustering

Let N points $\{x_i,\ldots,x_N\}$ be drawn independently from β distributions with mean μ_r where $r\in[\beta]$. Let number of samples drawn from distribution with mean μ_r be denoted by n_r and the separability condition (4.1) is satisfied. Additional notation used is introduced in Table A.1.

We now present an additional separability condition from [30] which is useful in order to prove some clustering results. For any $m \in [M]$ and $r, s \in [\beta]$,

$$|\mu(m,r) - \mu(m,s)| \geq c\phi_*(\frac{1}{n_s} + \frac{1}{n_r}), \tag{A.1}$$

where $\phi_* = \sum_i |x_i - \mathbb{E}(x_i)|$ and c is a constant.

Table A.1: Notation.

Δ_s	$\|\mu_s - \nu_s\|$
γ	$\max_{s,r\neq s} \frac{\Delta_s}{\|\mu_r - \mu_s\|}$
$\{\mathcal{T}_s\}_{s\in[\beta]}$	True partition of the samples X
n_s	$\|\mathcal{T}_s\|$
ϕ_*	$\sum_i \|x_i - \mathbb{E}(x_i)\|$
$g(S)$	$\frac{1}{\|S\|}\sum_{i\in S} x_i$
ρ^s_{in}	Fraction of points misclassified as cluster s $\frac{\sum_{r\neq s}\|\mathcal{T}_r\cap S_s\|}{n_s}$
ρ^s_{out}	Fraction of misclassified points in cluster s $\frac{\sum_{r\neq s}\|\mathcal{T}_s\cap S_r\|}{n_s}$

We first present the following lemma which describes the relationship between the separability conditions (4.1) and (A.1).

Lemma 7. *If the separability condition* (4.1) *is satisfied and* $N = T_0 = \frac{32\exp(\frac{K-1}{M-1})M}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}$, *then for any* r, s, *with high probability*

$$|\mu_r - \mu_s| \geq c\phi_*(\frac{1}{n_s} + \frac{1}{n_r}), \tag{A.2}$$

where $\phi_* = \sum_i |x_i - \mathbb{E}(x_i)|$ *and* c *is a constant.*

Proof. If suffices to show that with high probability,

$$4M\exp(\frac{K-1}{M-1})\sqrt{\sigma^2 + \epsilon_2} \geq (\frac{1}{n_r} + \frac{1}{n_s})\sum_{i\in[N]}|x_i - E[x_i]|.$$

From Hoeffding's inequality, we have

$$\Pr(\frac{1}{N}\sum_{i\in[N]}(x_i - E[x_i])^2 - \sigma^2 \geq \epsilon_2) \leq \exp(-2N\epsilon_2^2),$$

i.e., with probability greater than $1 - \frac{\delta}{2MK\beta(\beta+1)}$, we have $\sum_{i\in[N]}(x_i - E[x_i])^2 \leq N(\sigma^2 + \epsilon_2)$.

We have $||x||_1 \leq \sqrt{N}||x||_2$.

$$\begin{aligned}(\frac{1}{n_r}+\frac{1}{n_s})\sum_{i\in[N]}|x_i-E[x_i]| &\leq (\frac{1}{n_r}+\frac{1}{n_s})\sqrt{N}\sqrt{\sum_{i\in[N]}(x_i-E[x_i])^2} \\ &\leq (\frac{1}{n_r}+\frac{1}{n_s})N\sqrt{\sigma^2+\epsilon_2} \\ &\leq 4M\exp(\frac{K-1}{M-1})\sqrt{\sigma^2+\epsilon_2}\,,\end{aligned}$$

where the last inequality follows because from Lemma 6, we have $n_s \geq \frac{16}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}$.

□

We now present some lemmas that are useful for proving that after clustering, the centroids are closer to the means of the distributions from which they are drawn.

Lemma 8. *If the separability condition* (A.1) *is satisfied, then after using Cluster algorithm, we have that for any fixed* ϵ, δ *and* $n_s \geq N_{\epsilon,\delta} = \lceil \frac{16}{\epsilon^2}\ln(\frac{\beta}{\delta})\rceil$, *with probability greater than* $1-\delta$,

$$|\hat{\mu}_s - \mu_s| = |g(S_s) - \mu_s| \leq \epsilon.$$

Proof. From Lemma 9, after the α approximation algorithm, we have $\Delta_s \leq 2(\alpha+1)\frac{\phi_*}{n_s}$ and $\gamma < \frac{2(\alpha+1)}{c}$. We need $\gamma \leq \frac{1}{8}$ which gives $a < \frac{c}{16} - 1$. From Lemma 11, $\rho^s_{in} + \rho^s_{out} \leq \frac{8}{c}$ which we need to be less than $\frac{1}{2}$ thus giving us $c > 16$. From this and Lemma 10, the conditions for Lemma 12 are satisfied and $\gamma < \frac{1}{8}$. Thus we have

$$|g(S_s)-\mu_s| \leq 2(1-\rho^s_{out})|g(S_s\cap\mathcal{T}_s)-\mu_s| + 4\sum_{r\neq s:\rho^s_{in}(r)\neq 0}\rho^s_{in}(r)|g(S_s\cap\mathcal{T}_r)-\mu_r|.$$

For each $r \in [\beta]$, $S_s \cap T_r$ denotes independently drawn bounded random variables from reward distribution with mean μ_r.

$$\begin{aligned}
\Pr(\exists r \text{ s. t } |g(S_s \cap T_r) - \mu_r| \geq \epsilon) &\leq \sum_{r \in \beta} \Pr(|g(S_s \cap T_r) - \mu_r| \geq \epsilon) \\
&\leq_{(a)} \exp(-2n_s(1 - \rho^s_{out})(\frac{\epsilon}{4})^2) + \\
&\quad \sum_{r \neq s: \rho^s_{in}(r) \neq 0} \exp(-2n_s \rho^s_{in}(r)(\frac{\epsilon}{4})^2) \\
&\leq_{(b)} \exp(-2n_s \rho^s_{in}(\frac{\epsilon}{4})^2) + \\
&\quad \sum_{r \neq s: \rho^s_{in}(r) \neq 0} \exp(-2n_s c_1(\frac{\epsilon}{4})^2) \\
&\leq_{(c)} \beta \exp(-2n_s c_1(\frac{\epsilon}{4})^2) \leq \delta,
\end{aligned}$$

where $c_1 = \min_{r,s} \rho^s_{in}(r) : \rho^s_{in}(r) \neq 0$. Inequality (a) follows from Hoeffding's lemma, inequality (b) from $1 - \rho^s_{out} \geq \rho^s_{in}$ and inequality (c) from the definition of c_1.

For $\beta \exp(-2n_s c_1(\frac{\epsilon}{4})^2)) \leq \delta$, we need $n_s \geq \frac{8}{c_1 \epsilon^2} \ln(\frac{\beta}{\delta})$. Since $c_1 < \frac{1}{2}$, we have

$$n_s \geq \frac{16}{\epsilon^2} \ln(\frac{\beta}{\delta}).$$

Thus, with probability greater than $1 - \delta$, we have

$$|g(S_s) - \mu_s| \leq 2\frac{\epsilon}{4} + 4 \sum_{r \neq s: \rho^s_{in}(r) \neq 0} \rho^s_{in}(r)\frac{\epsilon}{4} \leq \frac{\epsilon}{2} + \rho^s_{in}\epsilon \leq \epsilon.$$

$\square$

Lemma 9. *An α approximation algorithm returns the set of centroids $\{\nu_1, \ldots, \nu_\beta\}$ where $C(x)$ returns the centroid of the cluster to which x belongs. We have $\forall \mathcal{T}_s$ $\exists \nu_s$ such that $|\nu_s - \mu_s| \leq 2(\alpha + 1)\frac{\phi_*}{n_s}$ and $\gamma < \frac{2(\alpha+1)}{c}$.*

Proof. We first show that $\forall s$, $\Delta_s \leq (\alpha + 1)\frac{\phi_T}{n_s}$ where $\phi_T = \sum_{s=1}^{\beta} \sum_{x \in \mathcal{T}_s} |x - g(\mathcal{T}_s)|$. Assume the contrary that for some $\mathcal{T}_s$,$|\nu_r - \mu_s| > (\alpha + 1)\frac{\phi_T}{n_s}$ $\forall r \in [\beta]$.

$$\begin{aligned}
\sum_{x \in T_s} |x - C(x)| &\geq \sum_{x \in T_s} |C(x) - g(T_s)| - |x - g(T_s)| \\
&> |T_s| \frac{(\alpha + 1)\phi_T}{|Ts|} - \sum_{x \in T_s} |x - g(T_s)| \\
&\geq (\alpha + 1)\phi_T - \phi_T = a\phi_T,
\end{aligned}$$

which is a contradiction. We now show that $\phi_T \leq 2\phi_*$, which proves that $\Delta_s \leq 2(\alpha+1)\frac{\phi_*}{n_s}$.

$$\begin{aligned}
\phi_T &= \sum_{s=1}^{\beta}\sum_{x\in T_s}|x-g(T_s)| \\
&\leq \sum_{s=1}^{\beta}\sum_{x\in T_s}|g(T_s)-\mu_s| + \sum_{s=1}^{\beta}\sum_{x\in T_s}|x-\mu_s| \\
&= \sum_{s=1}^{\beta}|T_s||g(T_s)-\mu_s| + \phi_* \\
&= \sum_{s=1}^{\beta}|\sum_{x\in Ts} x-\mu_s| + \phi_* \\
&\leq \sum_{s=1}^{\beta}\sum_{x\in Ts}|x-\mu_s| + \phi_* \\
&= 2\phi_*.
\end{aligned}$$

Now we show that $\gamma \leq \frac{2(\alpha+1)}{c}$. For any s, r,

$$\frac{2(\alpha+1)}{c}|\mu_r-\mu_s| \geq \frac{2(\alpha+1)}{c}c\phi_*(\frac{1}{n_s}+\frac{1}{n_r}) \geq \Delta_s.$$

Since this is true for all r, s, we have

$$\gamma \leq \frac{2(\alpha+1)}{c}.$$

□

Lemma 10. *If $\gamma < \frac{1}{4}$, the following results hold $\forall x \in S_r$,*

1. $|x-\mu_s| \geq (\frac{1}{2}-2\gamma)|\mu_r-\mu_s|, \quad \forall s \neq r.$
2. $|x-\mu_r| \leq \frac{1}{1-4\gamma}|x-\mu_s|.$

Proof. (1)

$$\begin{aligned}
|\nu_r-\nu_s| &= |\nu_r-\mu_r+\mu_r-\mu_s+\mu_s-\nu_s| \\
&\geq |\mu_r-\mu_s|-|\nu_r-\mu_r|-|\mu_s-\nu_s| \\
&\geq (1-2\gamma)|\mu_r-\mu_s|,
\end{aligned}$$

where the last inequality follows from the definition of γ.

$$\begin{aligned}
|x-\mu_s| &\geq |x-\nu_s| - |\mu_s-\nu_s| \\
&\geq \frac{1}{2}|\nu_r-\nu_s| - |\mu_s-\nu_s| \\
&\geq (\frac{1}{2}-\gamma)|\mu_r-\mu_s| - |\mu_s-\nu_s| \\
&\geq (\frac{1}{2}-\gamma)|\mu_r-\mu_s| - \gamma|\mu_r-\mu_s| \\
&= (\frac{1}{2}-2\gamma)|\mu_r-\mu_s|,
\end{aligned}$$

where the second inequality follows from $x \in S_r$ and the last from the definition of γ.

(2)

$$\begin{aligned}
|x-\mu_r| &\leq |\mu_r-\nu_r| + |x-\nu_r| \\
&\leq |\mu_r-\nu_r| + |x-\nu_s| \\
&\leq |\mu_r-\nu_r| + |x-\mu_s| + |\mu_s-\nu_s|.
\end{aligned}$$

Note that the first statement with the definition of γ also implies for $l = r, s$

$$\frac{1-4\gamma}{2\gamma}|\mu_l-\nu_l| \leq |x-\mu_s|,$$

which gives us

$$\begin{aligned}
|x-\mu_r| &\leq (1+\frac{4\gamma}{1-4\gamma})|x-\mu_s| \\
&= \frac{1}{1-4\gamma}|x-\mu_s|.
\end{aligned}$$

□

Lemma 11. *If* $\gamma < \frac{1}{4}$ *and* $|\mu_r - \mu_s| \geq c\frac{\phi_*}{n_s}$, *we have* $\rho^s_{in} \leq \frac{2}{(1-4\gamma)c}$ *and* $\rho^s_{out} \leq \frac{2}{(1-4\gamma)c}$.

Proof. From the separability condition (A.1), we have $|\mu_r - \mu_s| \geq c\frac{\phi_*}{n_s}$.

$$\begin{aligned} n_s\rho^s_{out}(\frac{1}{2} - 2\gamma)c\frac{\phi_*}{n_s} &\leq \sum_{r\neq s}|T_s \cap S_r|(\frac{1}{2} - 2\gamma)|\mu_s - \mu_r| \\ &\leq \sum_{r\neq s}\sum_{x_i \in T_s \cap S_r}(\frac{1}{2} - 2\gamma)|\mu_s - \mu_r| \\ &\leq \sum_{r\neq s}\sum_{x_i \in T_s \cap S_r}|x_i - \mu_s| \\ &\leq \phi_*, \end{aligned}$$

where the first and second inequalities follow from the separability condition and Lemma 10 respectively. This gives us $\rho^s_{out} \leq \frac{2}{(1-4\gamma)c}$ and similarly we also have $\rho^s_{in} \leq \frac{2}{(1-4\gamma)c}$. □

Lemma 12. *If (a)$\rho^s_{in} + \rho^s_{out} < \frac{1}{2}$ and (b)$|g(S_s \cap \mathcal{T}_r) - \mu_r| \geq (1-4\gamma)|g(S_s \cap \mathcal{T}_r) - \mu_s|$ we have*

$$|g(S_s) - \mu_s| \leq 2(1-\rho^s_{out})|g(S_s \cap \mathcal{T}_s) - \mu_s| + \frac{2}{1-4\gamma}\sum_{r\neq s}\rho^s_{in}(r)|g(S_s \cup \mathcal{T}_r) - \mu_r|.$$

Proof. $|g(S_s) - \mu_s|$

$$\begin{aligned} &= |\frac{|S_s \cap \mathcal{T}_s|g(S_s \cap \mathcal{T}_s) + \sum_{r\neq s}|S_s \cap \mathcal{T}_r|g(S_s \cap \mathcal{T}_r)}{|S_s|} - \mu_s| \\ &= \frac{|n_s(1-\rho^s_{out})(g(S_s \cap \mathcal{T}_s) - \mu_s) + \sum_{r\neq s} n_s\rho^s_{in}(r)(g(S_s \cap \mathcal{T}_r) - \mu_s)}{|S_s|}| \\ &\underset{(a)}{\leq} 2(1-\rho^s_{out})|(g(S_s \cap \mathcal{T}_s) - \mu_s)| + 2\sum_{r\neq s} n_s\rho^s_{in}(r)|(g(S_s \cap \mathcal{T}_r) - \mu_s)| \\ &\leq 2[(1-\rho^s_{out})|(g(S_s \cap \mathcal{T}_s) - \mu_s)| + \sum_{r\neq s} n_s\rho^s_{in}(r)|(g(S_s \cap \mathcal{T}_r) - \mu_s)|] \\ &\leq_{(b)} 2(1-\rho^s_{out})|(g(S_s \cap \mathcal{T}_s) - \mu_s)| + \frac{2}{1-4\gamma}\sum_{r\neq s} n_s\rho^s_{in}(r)|(g(S_s \cap \mathcal{T}_r) - \mu_r)|. \end{aligned}$$

□

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